

May 24, 1966

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3,252,336

FOURIER TYPE MECHANICAL AMPLITUDE TRANSFORMERS

Filed Jan. 27, 1964

2 Sheets-Sheet 1

FIG. 1
PRIOR APPLICATION S.N.-267718

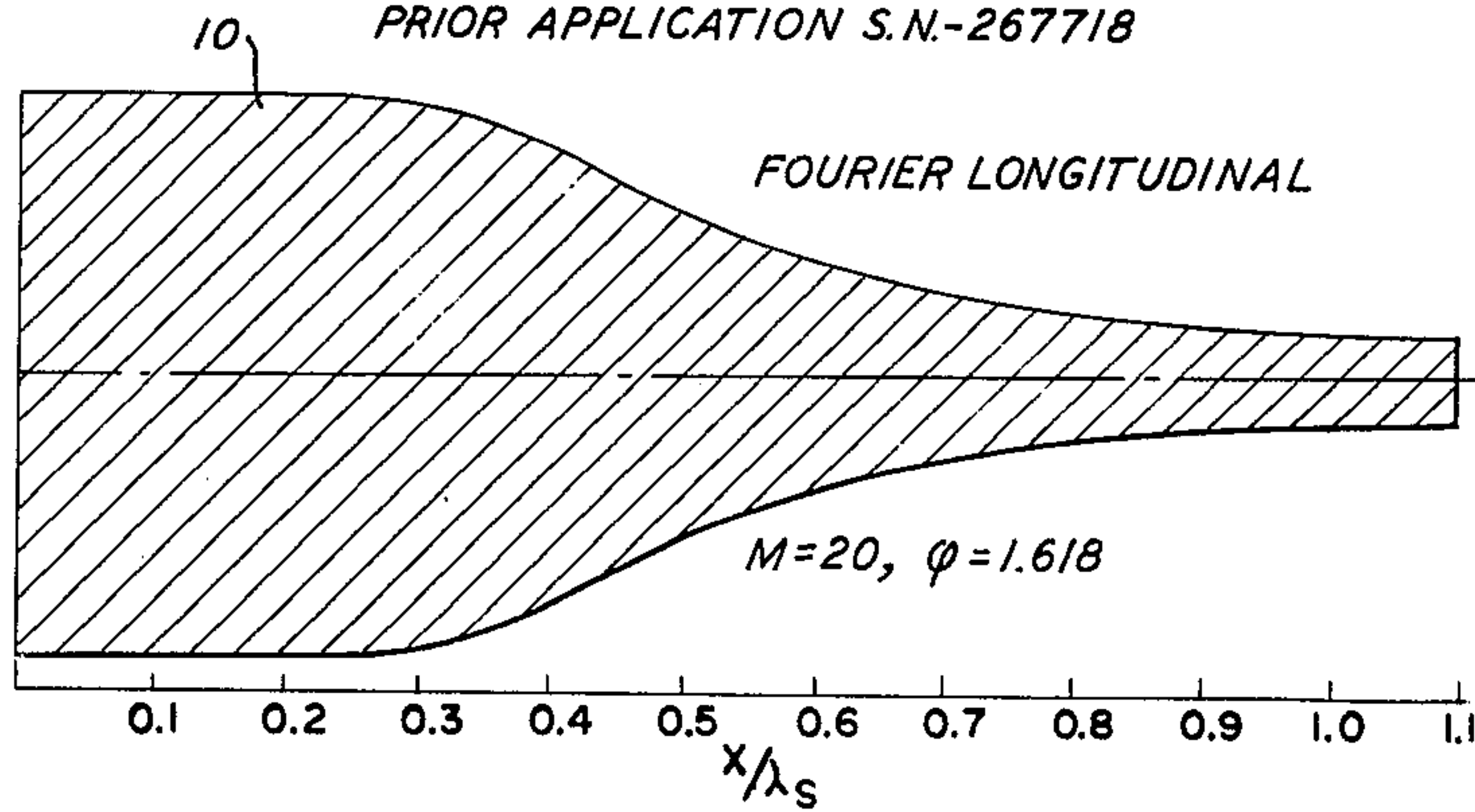


FIG. 2
FOURIER LONGITUDINAL

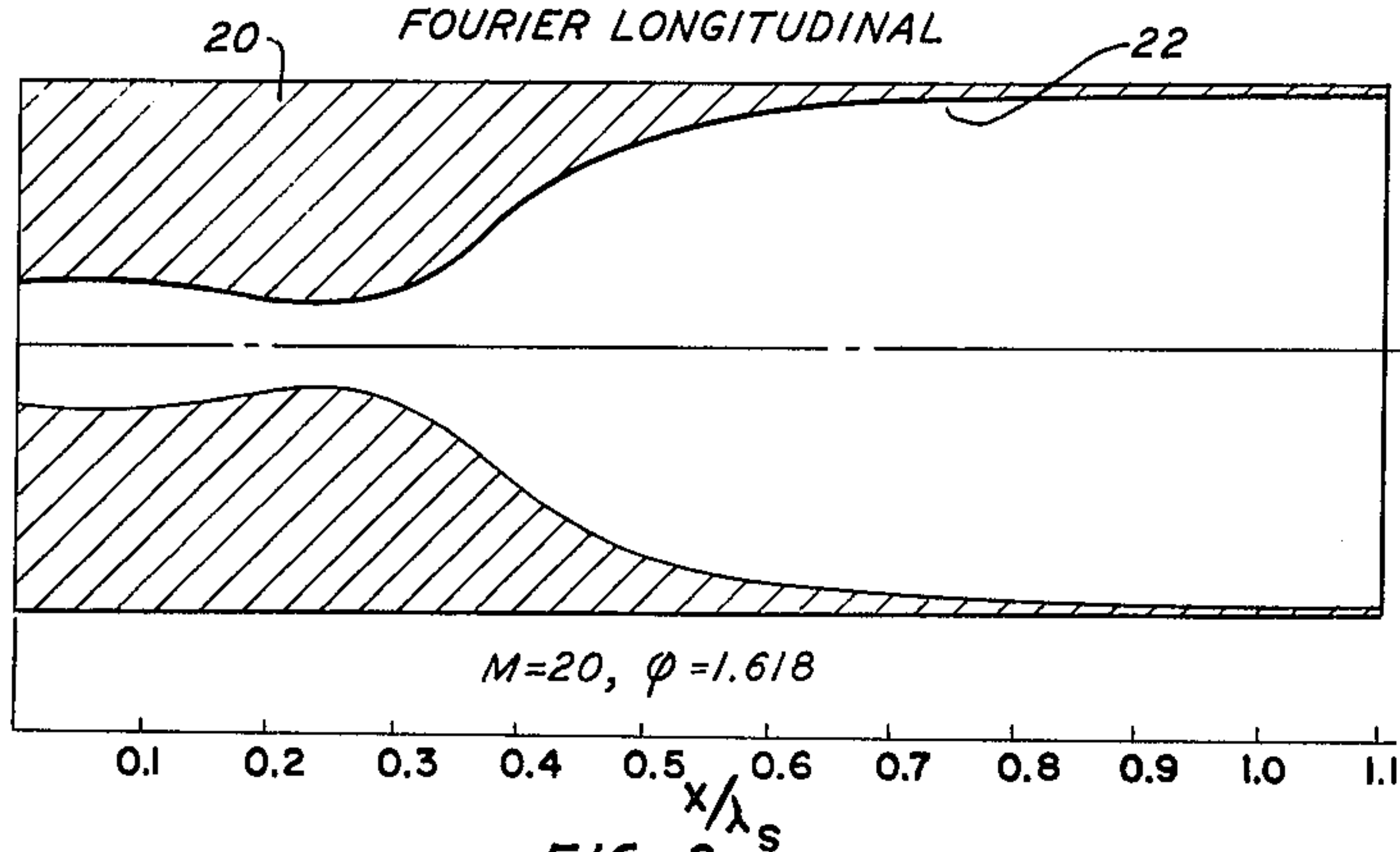
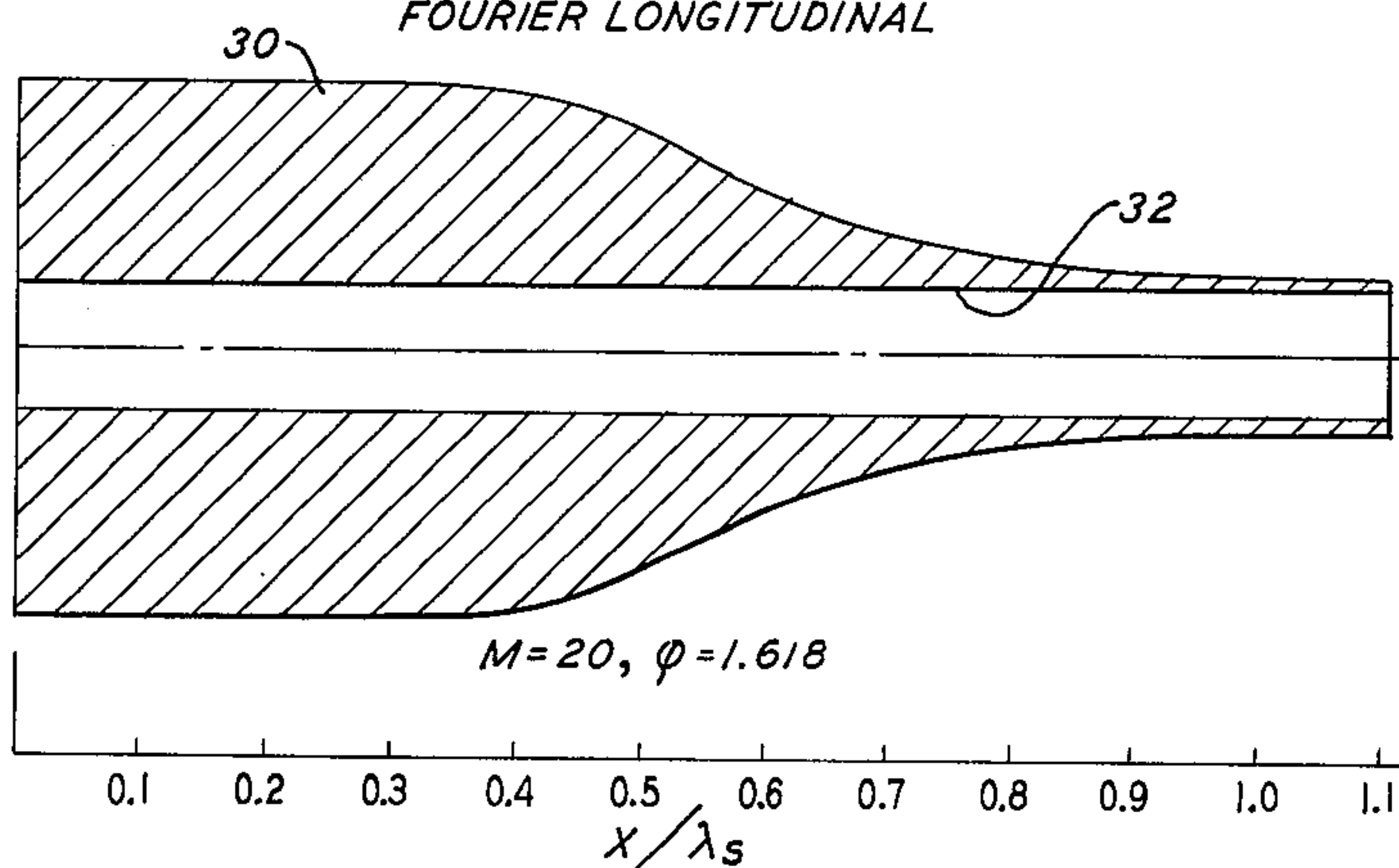


FIG. 3
FOURIER LONGITUDINAL



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May 24, 1966

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FOURIER TYPE MECHANICAL AMPLITUDE TRANSFORMERS

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FIG. 4
PRIOR APPLICATION S.N.-313628

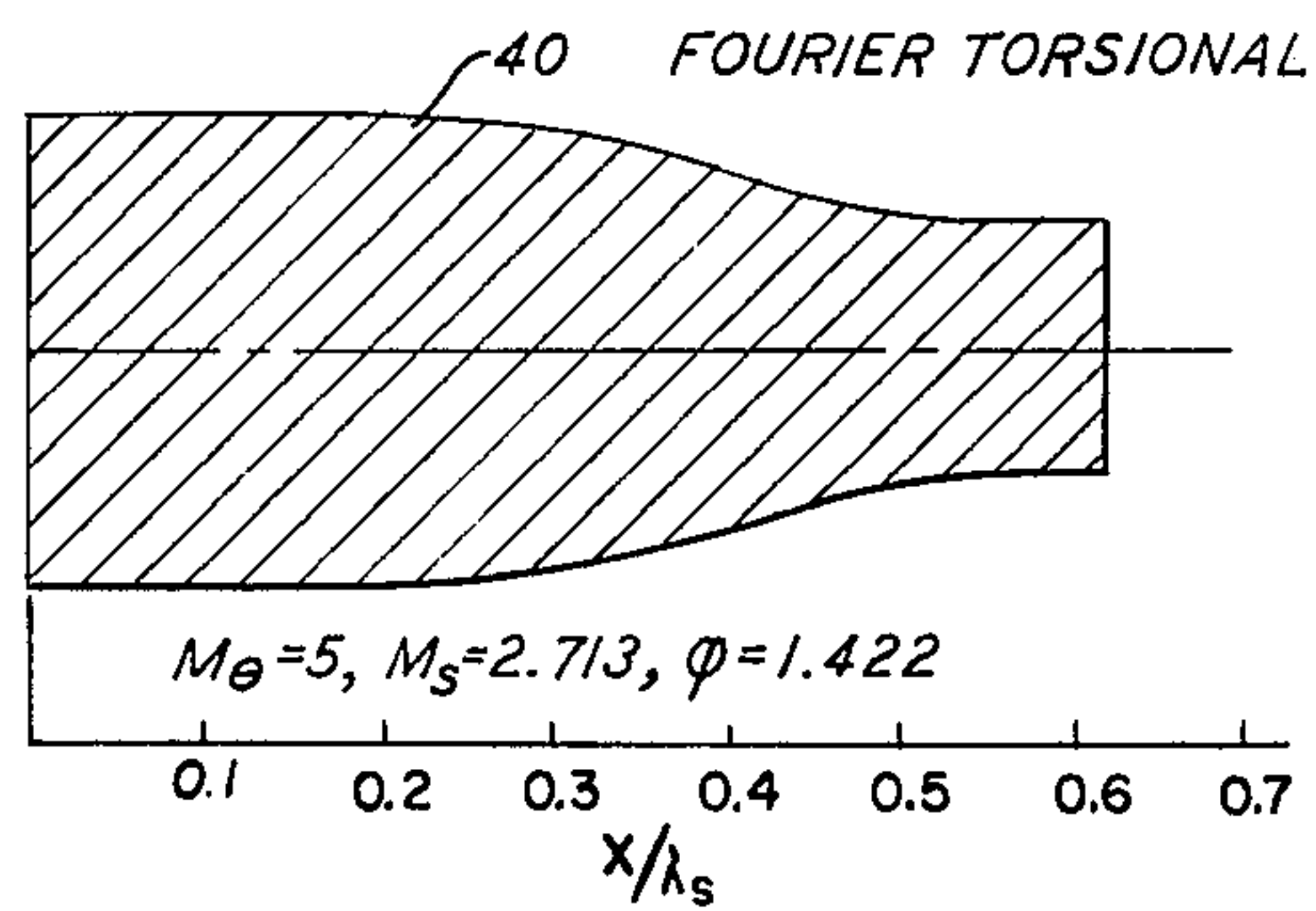


FIG. 5
FOURIER TORSIONAL

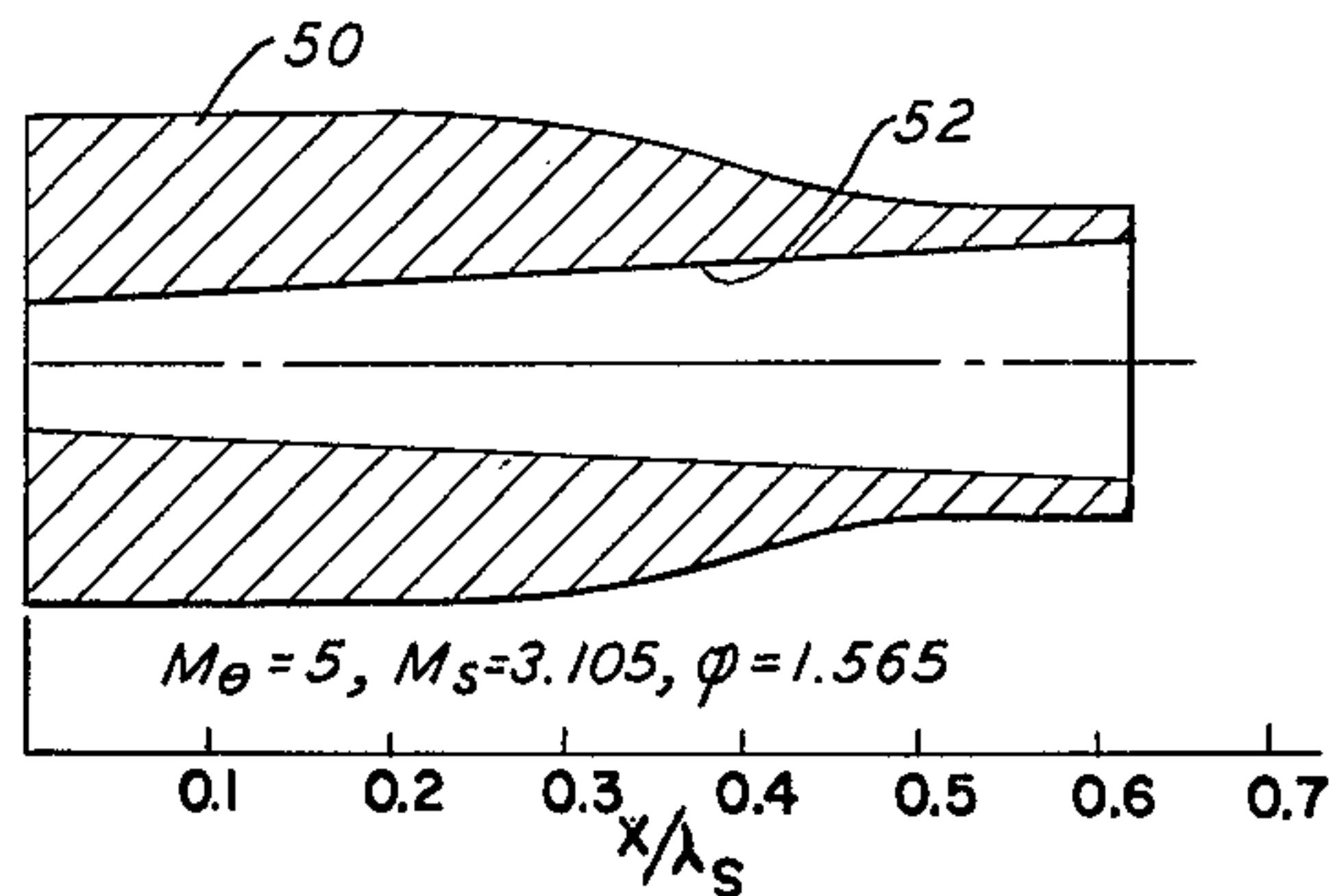
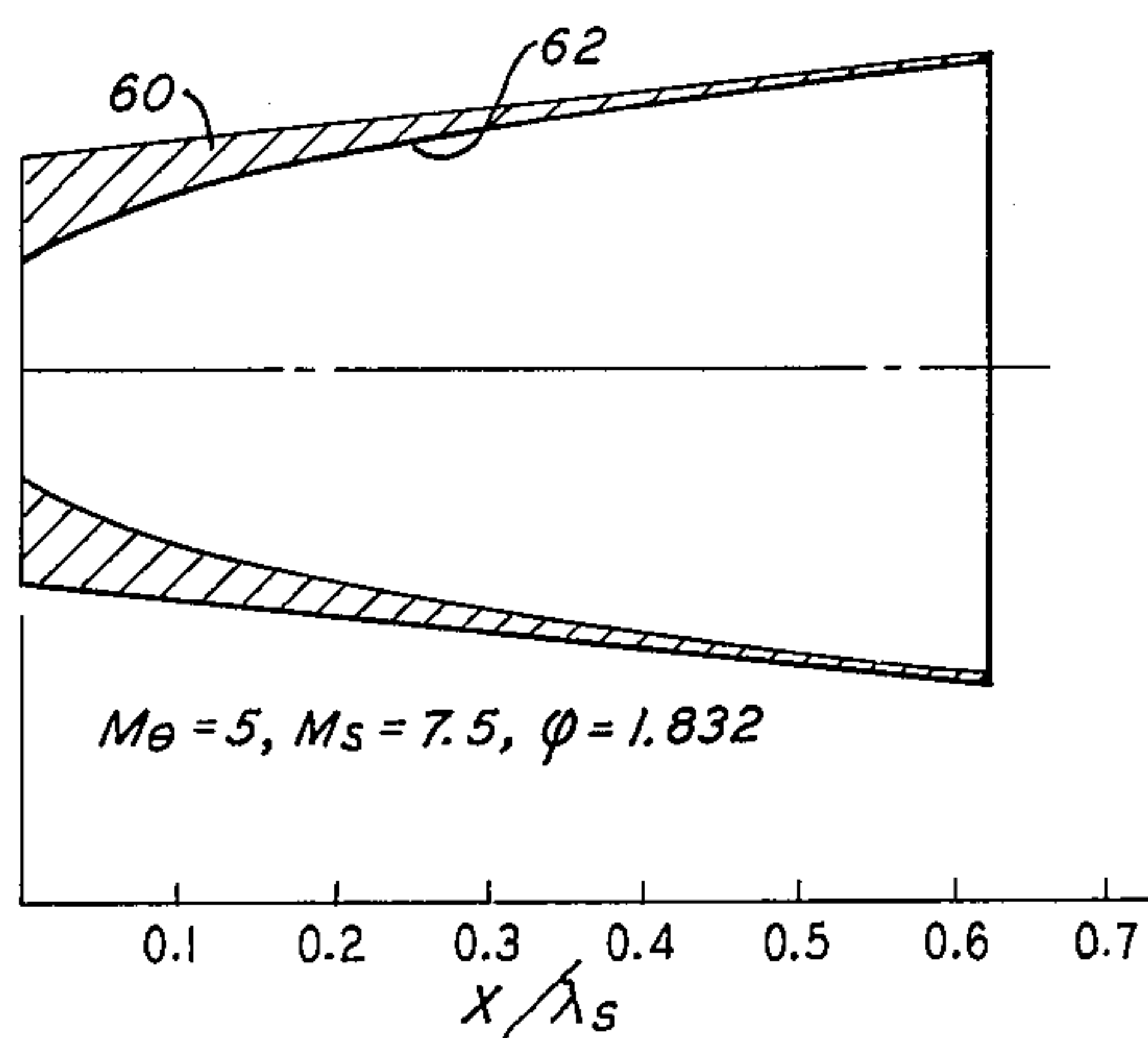


FIG. 6
FOURIER TORSIONAL



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FOURIER TYPE MECHANICAL AMPLITUDE TRANSFORMERS

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Filed Jan. 27, 1964, Ser. No. 340,453

5 Claims. (Cl. 74-1)

This invention relates to mechanical amplitude transformers comprising improved elongated mechanical vibratory members the longitudinal profiles or contours of which are derived from truncated Fourier series. More particularly, it relates to members of the above type having improved characteristics such as, for example, greater stiffness and/or higher figures of merit.

This application is a continuation-in-part of applicant's two copending applications, Serial No. 313,628, filed October 3, 1963, entitled, "Torsional Wave, Fourier Type, Mechanical Amplitude Transformer," and Serial No. 267,718, filed March 25, 1963, now United States Patent No. 3,175,406, entitled, "Mechanical Amplitude Transformer" (longitudinal vibratory, Fourier type). Both of these copending applications are assigned to applicant's present assignee and insofar as they are pertinent they are incorporated herein by reference and made an integral part of the present application.

Both of the aforesaid copending applications relate to elongated Fourier type mechanical amplitude transformers each of which may, for example, be conveniently made from a solid rod of resilient material such as steel, bronze, or the like, the external longitudinal profile or contour of which rod is varied in accordance with a function derived from a truncated Fourier series. The earlier filed of said copending applications relates to devices of the above described type designed for use with longitudinally vibratory energy while the later filed application relates to devices designed for use with torsionally vibratory energy.

The present application discloses how various properties, such as stiffness, figure of merit, amplification and the like, of devices of either or both of the aforesaid copending applications can be accentuated by the removal of material from the interior of the solid rod, instead of or in addition to modifications of the external longitudinal profile or contour so that for longitudinal vibratory energy the successive cross-sectional areas and for torsional vibratory energy the torsional rigidities of successive cross sections of the device vary in accordance with a function derived from a truncated Fourier series.

Some members of the present invention may have, with respect to comparable members of applicant's above-mentioned copending applications, a figure of merit (designated ϕ) which may be up to fifty percent greater or a stiffness which may be many times greater. Thus, for example, appropriately designed members of the present invention may be capable of transmitting a greatly increased amount of power without damage to the members than comparable members of my prior applications which latter were in turn very materially superior to comparable prior art members in this respect.

The above described improvements of the characteristics will obviously enhance the effectiveness of members of the present invention as practical tools for use in such

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operations as welding, drilling, agitating and the like, as compared with members of applicant's said copending applications which latter, as mentioned above, are materially superior in a number of respects to prior art members.

A principal object of the invention is, accordingly, to improve the operational characteristics of longitudinally tapered mechanical amplitude transformers of Fourier type longitudinal profile or contour.

This application further discloses that, by deriving profiles from truncated Fourier wave functions other than those used in the copending applications, vibrators having other operational characteristics may be produced.

The applications further disclose that vibrators with a great variety of cross-sectional shapes may be designed to have given operational characteristics, and, in particular, may have their profiles derived from truncated Fourier wave functions. The effect of varying the shapes of successive cross-sectional segments of a Fourier vibrator is discussed in my above-mentioned application Serial No. 313,628 at page 7, lines 15 through 29, inclusive, but as indicated at page 8, lines 1 through 8, inclusive, of said application, an arbitrary irrational variation of cross-sectional shape is not likely to be employed as a practical matter. However, circular, square, triangular or other regular cross-sectional shapes can obviously be readily employed for Fourier amplitude transformers.

Additional objects, features and advantages of the invention will become apparent from a perusal of the following general mathematical analysis and the detailed descriptions of illustrative embodiments of the invention taken in conjunction with the accompanying drawing in which:

FIG. 1 illustrates for purposes of comparison a longitudinally vibratory member designed as taught in my above-mentioned copending application Serial No. 267,718;

FIG. 2 illustrates a first arrangement employing the principles of the present invention to alter and improve particular characteristics of the member of FIG. 1;

FIG. 3 illustrates a second arrangement employing the principles of the present invention to alter and improve particular characteristics of the member of FIG. 1;

FIG. 4 illustrates for purposes of comparison a torsionally vibratory member designed as taught in my above-mentioned copending application Serial No. 313,628;

FIG. 5 illustrates a first arrangement employing the principles of the present invention to alter and improve particular characteristics of the member of FIG. 4; and

FIG. 6 illustrates a second arrangement employing the principles of the present invention to alter and improve particular characteristics of the member of FIG. 4.

As pointed out, for example, in applicant's above-mentioned copending later filed application, Serial No. 313,628, at page 3, the equations governing the propagation of both longitudinal and torsional waves in thin tapered rods are of similar form.

Application Serial No. 313,628 then proceeds to develop, at page 6, the nondimensional or normalized Equation 6 for torsional waves and sets out beneath it the corresponding equation of applicant's above-mentioned first filed application, designated (7) of said later filed application, for longitudinal waves.

Similarly, the equivalent of Equation 14 at page 14 of applicant's above-mentioned first filed copending application, Serial No. 267,718, for longitudinal waves is de-

rived in the second application, Serial No. 313,628, for torsional waves and appears on page 10 as Equation 12g of the last mentioned application.

While applicant's above-mentioned applications disclosed only solid vibratory rod members in which tapering was effected by altering only the exterior longitudinal profiles or contours of the rod members, it has occurred to applicant that "tapering" or an equivalent effect can, in numerous instances, be equally well realized in whole or in part by the removal of appropriate portions of material from the interior of an initially solid rod of constant external size and shape. Indeed, as mentioned above, members formed in accordance with the principles of the present invention will, for longitudinal vibrators, have much greater flexural rigidity and, for torsional vibrators, substantially higher figures of merit and surface amplitude magnification than the previously disclosed members, and may therefore have substantially higher figures of merit (φ) and may be more effective for a number of important applications.

Accordingly, a generalized mathematical treatment of Fourier amplitude transformers including those of my above-mentioned two copending applications, those presented as illustrative in the present application and numerous and varied other Fourier amplitude transformers which will readily occur to those skilled in the art will now be presented.

General analysis of all Fourier amplitude transformers

The class of Fourier amplitude transformers, including those of my two above-mentioned copending applications and those of the present application, can be expressed broadly as follows:

Consider a rod of length l . The cross section may vary in shape and in size, subject to the following conditions:

(a) The centroids of the successive cross sections lie on a straight line (of length l);

(b) The elastic properties and the density are constant in any one section;

(c) The maximum dimension of each cross section is much smaller than the wavelength, λ , of waves of the kind to be considered, in a uniform rod of the elastic constants and density of the material of that cross section, say, smaller than $\lambda/3$;

(d) The longitudinal component of the surface-slope of the rod is never very large, say smaller than 45° .

The longitudinal and torsional elastic waves in the rod can be closely described by the relation

$$\frac{\partial}{\partial x} \left(C B_1 \frac{\partial \Psi}{\partial x} \right) = \rho B_2 \frac{\partial^2 \Psi}{\partial t^2} \quad (1)$$

where:

ρ is the density at coordinate x and D is a characteristic dimension of the cross section at coordinate x ;

and for longitudinal waves, at coordinate x ,

$\Psi = \xi =$ longitudinal displacement at time t ,

$C = E =$ Young's modulus,

$B_1 = B_2 = A =$ area of the section $= k_A D^2$;

while, for torsional waves, at coordinate x ,

$\Psi = \theta =$ torsional displacement at time t ,

$C = n =$ shear modulus,

$B_1 =$ "torsional rigidity" $= \frac{1}{n} (\text{couple}) / \frac{\partial \theta}{\partial x} = k_1 D^4$,

$B_2 =$ polar second moment of the section about the longitudinal axis $= k_2 D^4$.

If the motion is simply harmonic, with pulsance p , then

$$\Psi(x, t) = \psi(x) e^{i p t} \quad (2)$$

and

$$\frac{d^2 \psi}{dx^2} + \frac{d}{dx} [\log_e (C B_1)] \frac{d\psi}{dx} + \left(\frac{2\pi}{\lambda} \right)^2 \frac{B_2}{B_1} \psi = 0 \quad (3)$$

where

$$\lambda = \frac{2\pi}{\rho} \sqrt{\frac{C}{\rho}}$$

and is the wavelength of the appropriate kind of waves in a thin, uniform rod of this C and ρ . From this,

$$\frac{d}{dx} [\log_e (C B_1)] = - \frac{\frac{d^2 \psi}{dx^2} + \left(\frac{2\pi}{\lambda} \right)^2 \frac{B_2}{B_1} \psi}{\frac{d\psi}{dx}} \quad (3a)$$

B_2/B_1 is, for a given type of wave (that is, longitudinal or torsional), a function of cross-sectional shape only. Let

$$\left(\frac{2\pi}{\lambda} \right)^2 \frac{B_2}{B_1} = \omega \quad (4)$$

In general, ω is a function of x . Suppose $\omega(k)$ and $\psi(x)$ are known explicitly. Let

$$P = C(x) B_1(x) / [C(x_0) B_1(x_0)]$$

where x_0 is some chosen value of x . Then P may be thought of as the defining function for the longitudinal profile. For longitudinal vibrations

$$P = [E(x)/E(x_0)] \cdot [D(x)/D(x_0)]^2$$

For torsional vibrations

$$P = [k_1(x)/k_1(x_0)] \cdot [n(x)/n(x_0)] \cdot [D(x)/D(x_0)]^4$$

Now, the elastic modulus must be known as a function of x , and so, in the torsional case, must k_1 (which is a function of cross-sectional shape only); otherwise ω would not be known. Therefore, once $P(x)$ has been calculated, $D(x)/D(x_0)$ follows. The latter function expresses the longitudinal profile of the vibrator, although its shape will be the same as that of the profile only when all cross sections of the vibrator are similar in shape and orientation.

We have considered the case in which

$$\psi = \sum_{j=0}^N [\alpha_j \cos j\pi x/l + \epsilon_j \sin j\pi x/l] \quad (5)$$

Any form of profile defined by the above relation for P and a wave function of the type of Equation 5, linked by Equation 3a will be a "Fourier" vibrator. The coefficients α_j , ϵ_j have to satisfy certain conditions, and these will specify the type of vibrator considered.

We have considered, in detail, amplitude transformers with stress-free or quasi-stress-free ends and have considered only $N \leq 4$. If the arithmetic ratio of the amplitudes at the two ends of the transformer is M (the "magnification"), and if

$$\beta_j = \frac{32}{M-1} \alpha_j$$

$$\gamma = 2l/\lambda$$

$$\mu = (M+1)/(M-1)$$

then, in the simplest case,

$$\beta_0 = 3\beta_4 - 4(4 - \gamma^2)$$

$$\beta_1 = 2\mu(9 - \gamma^2)$$

$$\beta_2 = -4(\beta_4 + \gamma^2)$$

$$\beta_3 = 2\mu(\gamma^2 - 1)$$

and $\epsilon_j = 0$ for all j . $N < 4$ yields special cases of these relations.

With these values of the β 's, Equation 3a can be integrated, yielding the profile or contour function P , if the variation of $\omega(x)$ is known. We have considered only the simplest case of constant ω , and have derived the profiles of greatest interest. However, many other sets of

boundary conditions can be fitted with truncated Fourier series of type (5); any such resulting wave function, $\psi(x)$, combined with any known variation of $\omega(x)$, will yield, after integration of Equation 3a, the profile or contour function, $P(x)$, of a Fourier vibrator. If ω is known only as a function of CB_1 , the problem can still be solved, though with much greater complexity, to yield a Fourier vibrator.

As the generalized counterpart of Equation 14 of my above-mentioned first filed copending application, Serial No. 267,718, and Equation 12g of my second filed copending application, Serial No. 313,628, the following equation is presented:

$$\frac{d(\log_e Z)}{d(\cos \pi X)} = \frac{(4 + \Gamma^2)(\Gamma^2 + \Phi) - \mu(9 - \Gamma^2)(\Gamma^2 - 1)(\cos \pi X - (16 - \Gamma^2)\Phi \cos^2 \pi X)}{\mu(3 - \Gamma^2) - 2(\Gamma^2 + 2\Phi) \cos \pi X + 3\mu(\Gamma^2 - 1) \cos^2 \pi X + 4\Phi \cos^3 \pi X} \quad (6)$$

When the vibrator is designed to employ longitudinal vibratory energy, Z represents A , Γ represents γ , and Φ represents α_{42} of Equation 14 of said first filed application.

When the vibrator is designed to employ torsional vibratory energy, Z represents K_1 , Γ represents γ_T , and Φ represents β_4 of Equation 12g of said second filed application.

Turning now to the drawing, FIGS. 1, 2 and 3 show in longitudinal cross section Fourier type vibrating members which may be of circular, square, triangular, rectangular, or other regular shape of transverse cross section, the transverse cross-sectional shape being the same throughout the length of the member, the dimensions of successive cross sections changing proportionately as required to afford the appropriate areas as required by the prescribed Fourier contouring of the member. For example, if the transverse cross-sectional shape is circular, successive cross sections vary only in diameter, or, if square, they vary only in the side dimensions, all cross sections having a common straight longitudinal axis and being aligned in the same manner with respect thereto. All of these members are designed to employ longitudinally vibratory energy.

The member 10 of FIG. 1 is a solid member designed as taught in my above-mentioned copending application Serial No. 267,718 and is included for the purpose of convenient comparison with the members 20 and 30 of FIGS. 2 and 3, respectively, which latter members illustrate applications of the principles of the present invention.

Incidentally, the member of FIG. 1 is substantially identical to the member 40 of FIG. 5B of application Serial No. 267,718 and FIGS. 5A and 5C of the same application show comparative "stepped" and "exponential" members.

While all three of the members 10, 20 and 30 of FIGS. 1, 2 and 3, respectively, have their respective successive transverse areas varied to effect contouring longitudinally in accordance with Equation 14 of application Serial No. 267,718, or Equation 6 of the present application, and have the same magnification M of 20 (that is, the particle velocities of the thinner or right ends of all three members are twenty times those of their respective thicker or left ends) and same figure of merit ϕ of 1.618, the stiffness of the thinner (right) end of the member 20 of FIG. 2 is substantially 200 times that of the thinner (right) end of the member 10 of FIG. 1 and the stiffness of the thinner (right) end of the member 30 of FIG. 3 is substantially seven times that of the thinner (right) end of the member 10 of FIG. 1.

As shown in FIG. 2, member 20 has a constant outer transverse dimension throughout its length and all longitudinal contouring is effected by the removal of material from the interior, thus producing cavity 22 within the member. Since it is more difficult as a practical matter to achieve the desired contouring by removing material only from the interior of the member, an alternative form of member of the invention is illustrated in FIG. 3.

In member 30 of FIG. 3 a hole 32 of constant transverse dimensions, concentric with the longitudinal axis of the member, effects the removal of a substantial amount of material from the interior and longitudinal contouring is effected by suitably profiling the exterior of the member to provide the appropriate successive transverse cross-sectional areas as required for a longitudinally vibrating Fourier amplitude transformer in accordance with Equation 6 hereinabove.

An alternative way of achieving the desired longitudinal contouring of a member of the invention by removing varying amounts of material from both the interior and exterior of a member of the invention is illustrated for

a torsionally vibrating member 50 in FIG. 5 and will be described in detail hereinafter. An arrangement substantially the converse of that illustrated in FIG. 5 is illustrated in FIG. 6 and will also be described in detail hereinafter.

In members 30 and 50 of FIGS. 3 and 5, respectively, the simple characters of the internal cavities greatly facilitate manufacture while the more complex contouring is relegated to the exteriors of the members where it can be much more readily effected.

FIGS. 4, 5 and 6 show Fourier type vibrating members 40, 50 and 60, respectively, in longitudinal cross section. All of these members are designed to employ torsionally vibratory energy. The member 40 of FIG. 4 is a solid member designed as taught in my above-mentioned copending application Serial No. 313,628 and is included for the purpose of convenient comparison with the members 50 and 60 of FIGS. 5 and 6, respectively, which latter members illustrate specific applications of the principles of the present invention.

All three members have their respective successive transverse polar second moments varied to effect contouring longitudinally in accordance with Equation 12g of application Serial No. 313,628 or Equation 6 of the present application and have the same torsional magnification $M_\theta = 5$.

As noted on the respective figures, however, member 50 of FIG. 5 has a particle velocity magnification M_s of 3.105 and a figure of merit ϕ of 1.565 as compared with 2.713 and 1.422, respectively, for member 40 of FIG. 4.

Similarly, member 60 of FIG. 6 has a particle velocity magnification M_s of 7.5 and a figure of merit ϕ of 1.832.

The above noted increased values of the figure of merit ϕ for the members 50 and 60 of FIGS. 5 and 6, respectively, indicate that substantially greater amounts of power can be transmitted through the members without danger of damage to the members.

The above noted increased values of the particle velocity magnification M_s indicate that the members will be substantially more effective for such operations as welding, drilling and the like. This effectiveness is, of course, further enhanced by the above-mentioned fact that the members can transmit greater amounts of power.

Member 50 of FIG. 5 combines the tapering effects of a conical longitudinal cavity 52 with external tapering as illustrated on the drawing.

Member 60 of FIG. 6 combines the tapering effects of an external conical taper with internal tapering of the longitudinal cavity 62 within the member.

A magnification as low as $M_\theta = 5$ was chosen for the torsional members since the thinner or right end of members of the type illustrated in FIG. 6 would have become too thin to be represented in even approximately true proportion with respect to their thicker or left ends if members having higher magnifications had been illustrated.

The dimensions of a longitudinal vibrator are naturally related to the wavelength, λ_L , for longitudinal waves in a thin uniform rod, while those of a torsional vibrator

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are related to λ_s , the wavelength of shear waves. It can be shown that, for an isotropic material,

$$\lambda_L = [2(1+\nu)]^{1/2} \lambda_s$$

where ν is Poisson's ratio of the material. So that the members of FIGS. 1, 2 and 3 can be shown to a common longitudinal scale with the members of FIGS. 4, 5 and 6, assuming, of course, that they are all of the same material and have the same frequency of resonance, a common value of $\nu=0.3$ has been employed (approximately the value for steel).

From the above generalized mathematical analysis, taken in conjunction with the disclosures of applicant's two above-mentioned copending applications and the

$$\frac{d(\ln A)}{d(\cos \pi X)} = \frac{(4-\gamma^2)(\gamma^2+\alpha_{42})-\mu(9-\gamma^2)(\gamma^2-1) \cos \pi X - (16-\gamma^2)\alpha_{42} \cos^2 \pi X}{\mu(3-\gamma^2)-2(\gamma^2+2\alpha_{42}) \cos \pi X + 3\mu(\gamma^2-1) \cos^2 \pi X + 4\alpha_{42} \cos^3 \pi X}$$

disclosure of the present application, it is apparent that numerous and varied modifications and rearrangements of the novel illustrative specific structures disclosed can readily be devised by those skilled in the art without departing from the spirit and scope of applicant's invention. The disclosed structures are, accordingly, to be understood as being particular illustrative species of the invention only and in no way as limiting the same since obviously a very large class of distinctly differing Fourier type members can readily be derived by straightforward applications of the generalized analysis given hereinabove.

What is claimed is:

1. A mechanical transformer for magnifying the amplitude of a repetitive, substantially sinusoidal, physical displacement by a factor M , M having a value exceeding 2, said transformer comprising an elongated acoustically resonant member having a concentric longitudinal cavity extending between its ends, the transverse cross-

sectional area A of the member varying from a maximum at one end to a minimum at the other end so as to provide a cross-sectional area A of the member varying between the one end and the other end in accordance with the integral of the relation

$$\frac{d(\ln A)}{d(\cos \pi X)} = \frac{(4-\gamma^2)(\gamma^2+\alpha_{42})-\mu(9-\gamma^2)(\gamma^2-1) \cos \pi X - (16-\gamma^2)\alpha_{42} \cos^2 \pi X}{\mu(3-\gamma^2)-2(\gamma^2+2\alpha_{42}) \cos \pi X + 3\mu(\gamma^2-1) \cos^2 \pi X + 4\alpha_{42} \cos^3 \pi X}$$

where

$$\gamma = \frac{2l}{\lambda}, \mu = \frac{M+1}{M-1}, X = \frac{x}{l}$$

where x represents the distance along the member from its larger end of the cross section for which the area A is being instantly determined and α_{42} is a constant at choice having a value between minus one and plus one.

2. A mechanical transformer for magnifying the amplitude of a repetitive, substantially sinusoidal, physical displacement by a factor M , M having a value exceeding 2, said transformer comprising an elongated rod having a substantially constant transverse outer size throughout its length, the rod having material removed from its interior so as to provide a cross-sectional area A varying between one end and the other end in accordance with the integral of the relation

$$\frac{d(\ln A)}{d(\cos \pi X)} = \frac{(4-\gamma^2)(\gamma^2+\alpha_{42})-\mu(9-\gamma^2)(\gamma^2-1) \cos \pi X - (16-\gamma^2)\alpha_{42} \cos^2 \pi X}{\mu(3-\gamma^2)-2(\gamma^2+2\alpha_{42}) \cos \pi X + 3\mu(\gamma^2-1) \cos^2 \pi X + 4\alpha_{42} \cos^3 \pi X}$$

where

$$\gamma = \frac{2l}{\lambda}, \mu = \frac{M+1}{M-1}, X = \frac{x}{l}$$

$$\frac{d(\ln K_1)}{d(\cos \pi X)} = \frac{(4-\gamma_T^2)(\gamma_T^2+\beta_4)-\mu(9-\gamma_T^2)(\gamma_T^2-1) \cos \pi X - (16-\gamma_T^2)\beta_4 \cos^2 \pi X}{\mu(3-\gamma_T^2)-2(\gamma_T^2+2\beta_4) \cos \pi X + 3\mu(\gamma_T^2-1) \cos^2 \pi X + 4\beta_4 \cos^3 \pi X}$$

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where x represent the distance along the member from its larger end of the cross section for which the area A is being instantly determined and α_{42} is a constant at choice having a value between minus one and plus one.

3. A mechanical transformer for magnifying the amplitude of a repetitive, substantially sinusoidal, physical displacement by a factor M , M having a value exceeding 2, said transformer comprising an elongated rod having a concentric longitudinal cavity extending between its ends, the exterior of the rod being tapered in longitudinal profile or contour to provide a cross-sectional area A varying between one end and the other end in accordance with the integral of the relation

where

$$\gamma = \frac{2l}{\lambda}, \mu = \frac{M+1}{M-1}, X = \frac{x}{l}$$

where x represents the distance along the member from its larger end of the cross section for which the area A is being instantly determined and α_{42} is a constant at choice having a value between minus one and plus one.

4. A mechanical transformer for magnifying the amplitude of a repetitive, vibratory, torsional physical displacement by a factor M_θ , M_θ having a value exceeding 2, said transformer comprising an elongated acoustically resonant member having a concentric longitudinal cavity extending between its ends, the transverse cross-sectional area of the member varying from a maximum at one end to a minimum at the other end so as to provide a torsional rigidity K_1 varying between the one end and the other end in accordance with the integral of the relation

where

$$\gamma_T = \frac{2l_T}{\lambda_T}, \mu = \frac{M_\theta+1}{M_\theta-1}, X = \frac{x}{l_T}$$

where x represents the distance, along the member from its larger end, of the cross section for which the torsional rigidity K_1 is being instantly determined and β_4 is a constant at choice having a value between minus one and plus one.

5. A mechanical transformer for magnifying the amplitude of a repetitive, vibratory, torsional physical displacement by a factor M_θ , M_θ having a value exceeding 2, said transformer comprising an elongated rod having a concentric longitudinal cavity extending between its ends, both the exterior and interior surfaces of the rod being longitudinally tapered, the combined tapers resulting in a longitudinal contour of successive transverse cross-sectional areas along the rod which varies so as to produce a torsional rigidity K_1 of the member between

one end of the rod and the other in accordance with the integral of the relation

where

$\gamma_T = \frac{2l_T}{\lambda_T}, \mu = \frac{M_\theta + 1}{M_\theta - 1}, X = \frac{x}{l_T}$

where x represents the distance, along the member from its larger end, of the cross section for which the torsional rigidity K_1 is being instantly determined and β_4 is a constant at choice having a value between minus one and plus one.

References Cited by the Examiner

UNITED STATES PATENTS

5	2,541,944	2/1951	Slaymaker et al.	181—5	X
	2,748,298	5/1956	Calosi et al.	74—1	X
	2,834,158	5/1958	Petermann	74—1	X
	2,948,154	8/1960	Kleesattel	74—1	
	3,019,660	2/1962	Welkowitz	74—1	
	3,131,515	5/1964	Mason	73—71.5	X

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