

May 24, 1966

E. EISNER

3,252,335

TORSIONAL WAVE, FOURIER TYPE, MECHANICAL AMPLITUDE TRANSFORMER

Filed Oct. 3, 1963

3 Sheets-Sheet 1

FIG. 1

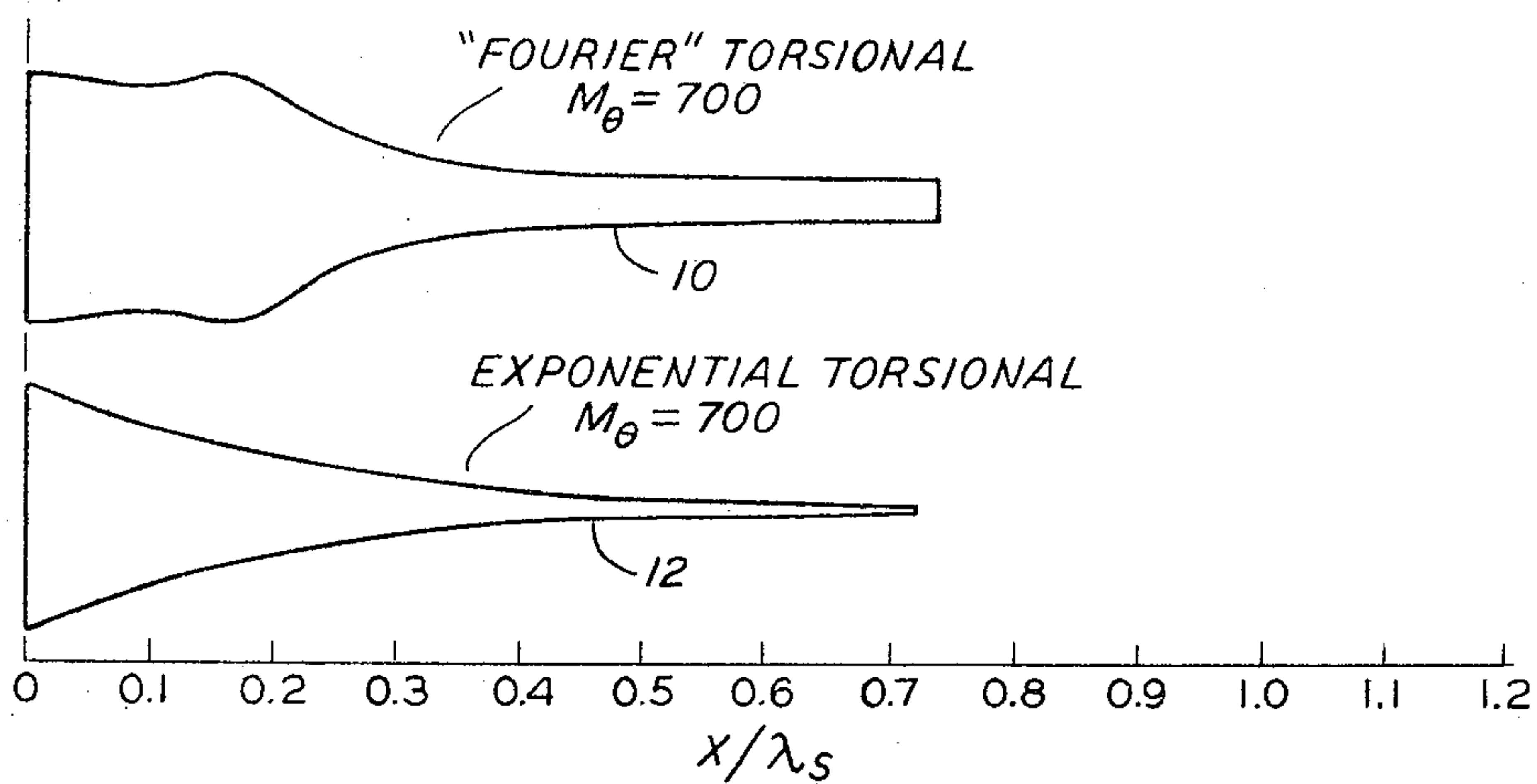
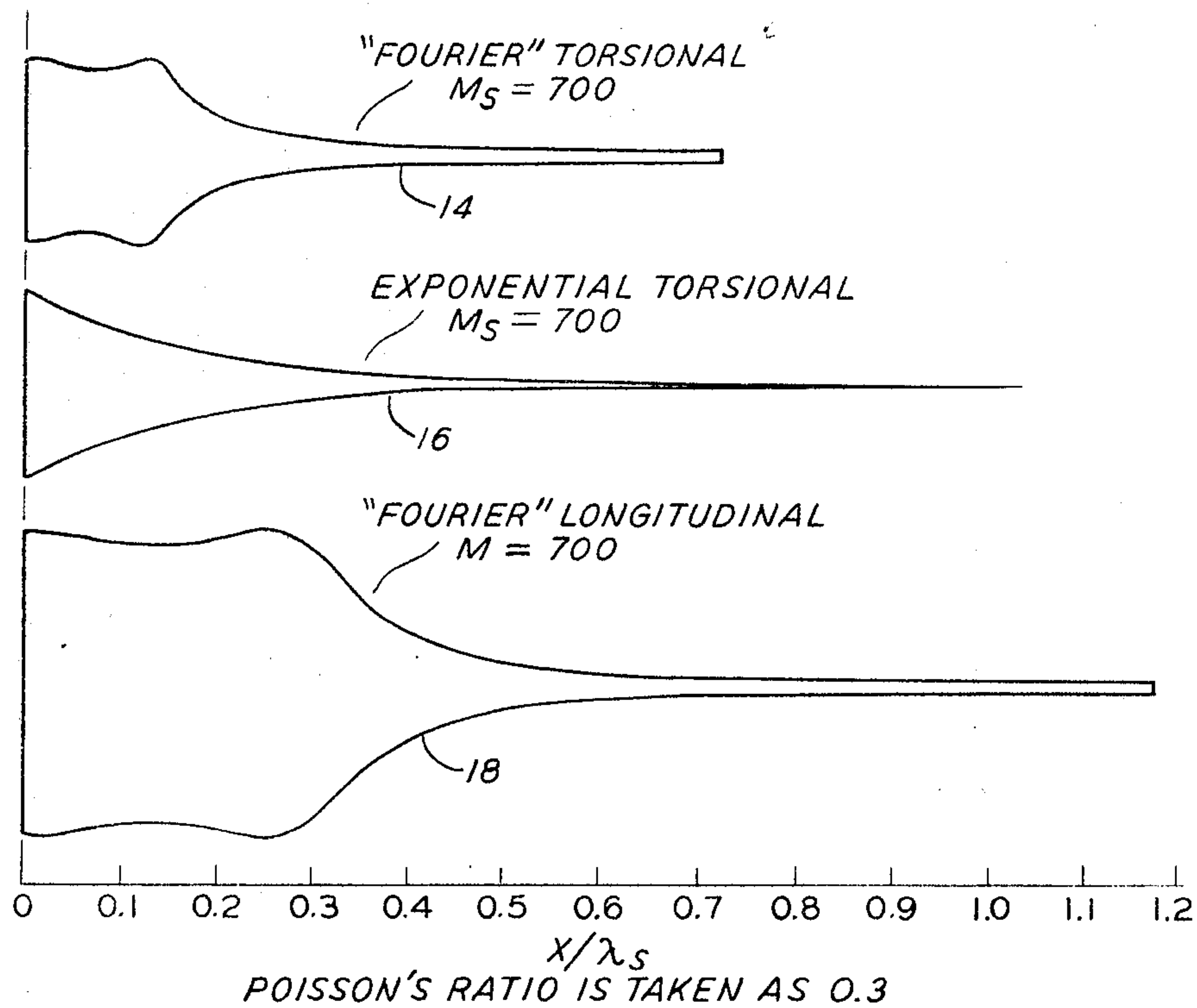


FIG. 2



INVENTOR
E. EISNER
BY *H. O. Wright*
ATTORNEY

May 24, 1966

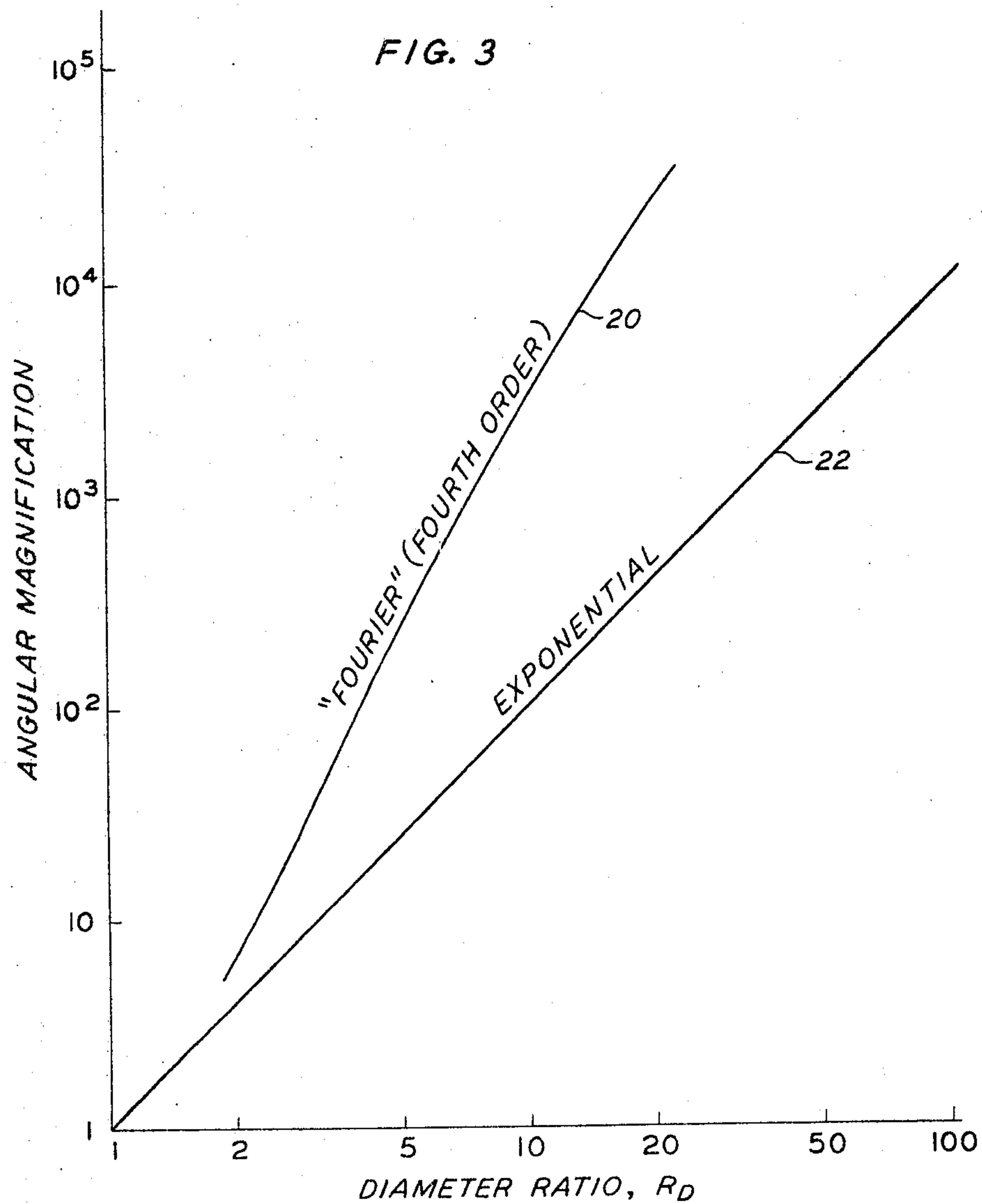
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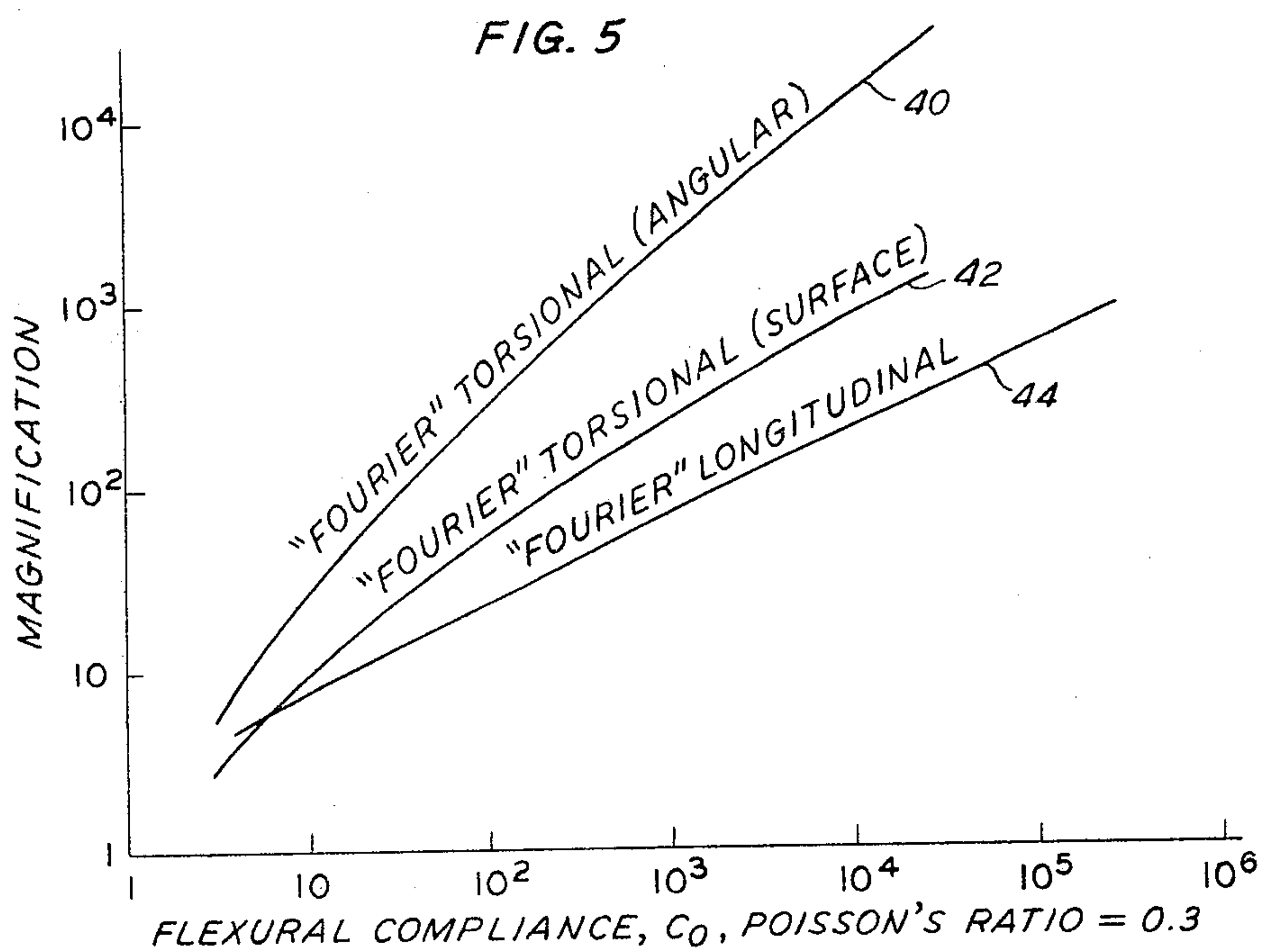
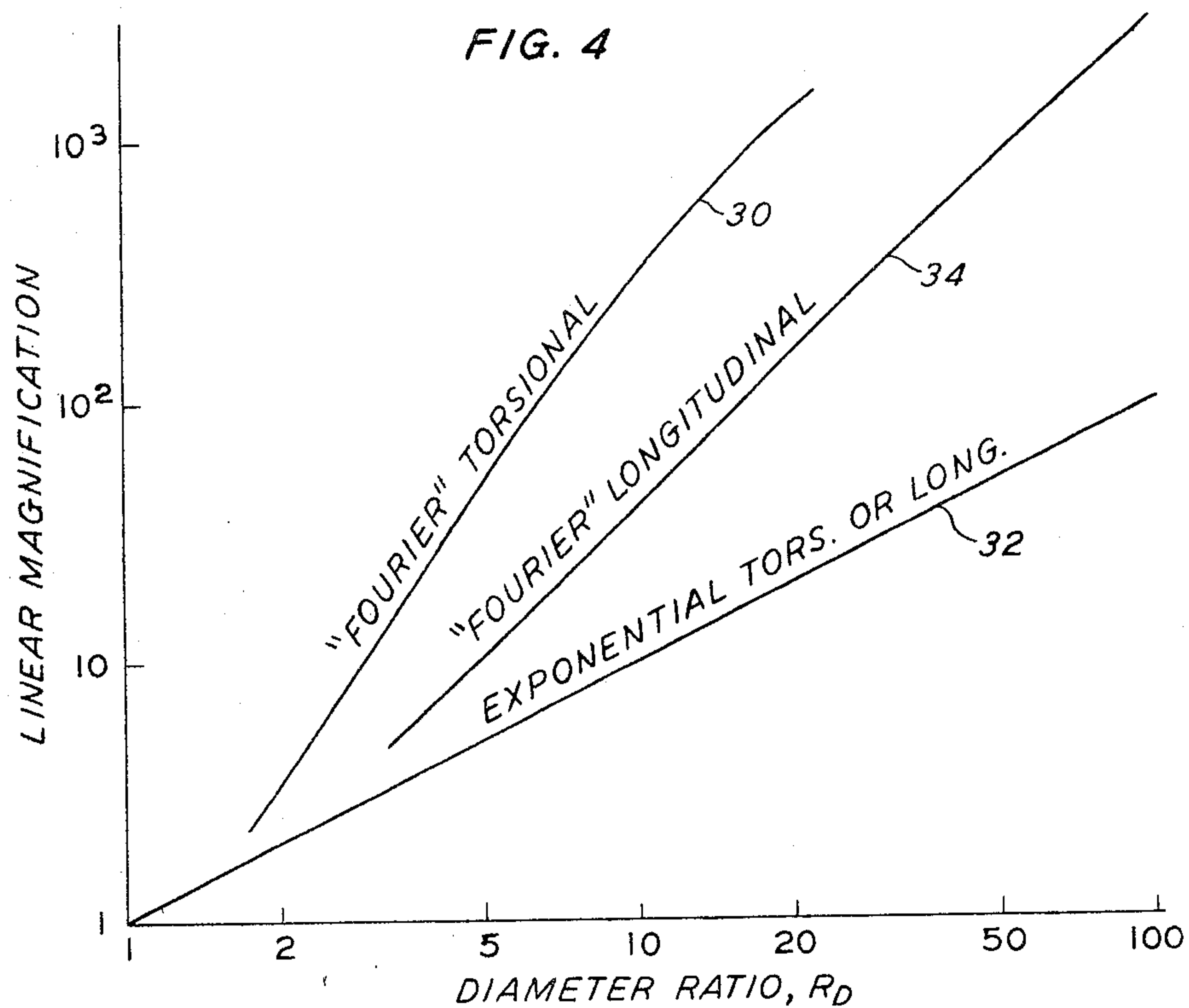
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TORSIONAL WAVE, FOURIER TYPE, MECHANICAL AMPLITUDE TRANSFORMER

Edward Eisner, Bernardsville, N.J., assignor to Bell Telephone Laboratories, Incorporated, New York, N.Y., a corporation of New York

Filed Oct. 3, 1963, Ser. No. 313,628

5 Claims. (Cl. 74-1)

This invention relates to elongated tapered mechanical vibratory members for increasing the magnitude of repetitive physical displacements. More particularly, it relates to members designed for use with torsional vibratory energy.

This application is a continuation-in-part of applicant's earlier copending application entitled, "Mechanical Amplitude Transformer," Serial No. 267,718, filed March 25, 1963, now United States Patent No. 3,175,406, and assigned to applicant's present assignee. The said earlier application discloses elongated amplitude transformers tapered in accordance with truncated Fourier series of the third and fourth orders to effect magnification of linear longitudinally vibratory motion. The present application extends the theory and practice of the said earlier application to "Fourier" tapered mechanical amplitude transformers for effecting the magnification of torsional vibratory motion. While the physical dimensions and longitudinal contours of devices of the present invention are distinctly different from those of devices adapted to use longitudinally vibratory energy of like frequency and magnification, the mathematical analysis and underlying theoretical aspects have a common basis as will become apparent hereinafter. Accordingly, applicant's abovementioned earlier application is incorporated by reference and made an integral portion of the present application insofar as it is pertinent.

It will be demonstrated hereinbelow that transverse vibrations of the type commonly known as torsional vibrations can be magnified approximately in direct proportion to the 3.5th power of the ratio of the maximum to the minimum diameters (R_D) of the member rather than being limited, as taught in the prior art for torsional exponentially tapered vibrators to the square (that is, the second power) of this ratio. Furthermore, the appropriate tapered longitudinal contours which afford optimum Fourier mechanical amplitude transformers of torsional vibratory energy from standpoints such as stiffness, and the like, will be presented.

Accordingly, a principal object of the present invention is to extend the capacity and utility of "Fourier mechanical amplitude transformers.

A further object is to remove limitations to the stability, ruggedness and ease of manufacture of mechanical amplitude transformers for torsional vibratory energy.

Other and further objects, features and advantages of the invention will become apparent from a perusal of the following detailed description of illustrative embodiments of the invention taken in conjunction with the accompanying drawing, wherein:

FIG. 1 illustrates the comparative longitudinal contours for "Fourier" Torsional and Exponential Torsional types of tapered vibrators both producing an angular magnification M_s of 700;

FIG. 2 illustrates the comparative longitudinal contours for "Fourier" Torsional, Exponential Torsional, and "Fourier" Longitudinal tapered vibrators, all three producing a linear magnification M_s of 700;

FIG. 3 illustrates the comparative angular magnifications versus maximum to minimum diameter ratio, R_D , for "Fourier" torsional and Exponential torsional types of tapered vibrators;

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FIG. 4 illustrates the comparative linear magnifications versus diameter ratio, R_D , for "Fourier" Torsional, "Fourier" Longitudinal and Exponential Torsional or Longitudinal types of tapered vibrators; and

FIG. 5 illustrates the comparative flexural compliances, C_0 , versus magnifications for "Fourier" Torsional (angular magnification), "Fourier" Torsional (surface motion magnification) and "Fourier" Longitudinal types of tapered vibrators.

The equations governing the propagation of longitudinal and torsional waves in thin, tapered rods are of similar form. It is therefore possible by the introduction of appropriate modifications to use substantially the method by which the "Fourier" type of longitudinally resonant amplitude transformers are designed, as taught in my abovementioned copending application, to design torsionally resonant transformers. These "Fourier" type torsional transformers are, however, capable of extremely high magnifications which cannot be realized by any practical structure designed in accordance with prior art teachings.

The stark contrast between comparable "Fourier" type torsional and exponentially tapered torsional transformer contours is graphically illustrated in FIG. 1 of the accompanying drawing in which member 10 is of "Fourier" type and member 12 is the comparable exponentially tapered type, both being designed for torsional vibrations of like frequency and like angular magnification.

In FIG. 2, the contrast between Fourier torsional, Fourier longitudinal and exponential torsional transformers is illustrated by the comparable contours, 14, 18, and 16, respectively, for like linear or particle velocity magnifications. A magnification of 700 is used for all the transformers in both FIGS. 1 and 2. Incidentally, the cross-sectional area of the left end of the exponential torsional transformer 16 is actually less than illustrated since sufficiently thin lines to illustrate the actual thickness dimension are not practicable for use in the drawing.

The equation governing the (one-dimensional) propagation of torsional waves in thin, straight rods of slowly varying section is readily found by considering the motion of a thin, cross-sectional slice. If θ is the angular displacement at coordinate x along the rod at time t , and n and ρ are the shear modulus and the density, respectively, of the (isotropic) material, then

$$\frac{\partial}{\partial x} \left(n K_1 \frac{\partial \theta}{\partial x} \right) = \rho K_2 \frac{\partial^2 \theta}{\partial t^2} \quad (1)$$

where

$$K_1 = k_1 D^4(x) \quad (1a)$$

$D(x)$ being a characteristic dimension of the cross section and k_1 , k_2 being nondimensional functions of cross-sectional shape only. K_1 is sometimes called the torsional rigidity of the section, while K_2 is the second moment of the section about the axis of torsion. K_2 is readily derived by simple integration, while K_1 is given for many shapes in engineering handbooks such as that entitled "Formulas for Stress and Strain" by R. J. Roark, published by McGraw-Hill Book Company, Inc., New York, 1954.

If the cross section of the rod gets too large, torsional modes not described by Equation 1 can be excited. W. P. Mason discusses this difficulty for the circular section type of vibrator in his book entitled "Physical Acoustics and the Properties of Solids," published by D. Van Nostrand Company, Inc., New York, 1958. The limitation on "diameter" is even more severe for section shapes that depart radically from the circular, such, for example, as narrow rectangles. Also, validity of Equation 1 requires that the angle of taper of the rod shall not be too great.

If waves of a single frequency, $p/2\pi$, where $p=2\pi f$, are considered, then

$$\Theta(x, t) = \theta(x) e^{ipt} \quad (2)$$

Also let

$$\lambda_s = \frac{2\pi}{p} \left(\frac{n}{\rho} \right)^{1/2} \quad (3)$$

so that λ_s is the wavelength of shear waves of frequency $p/2\pi$ in the material. If the shear modulus n is constant, then

$$\frac{d^2\theta}{dx^2} + \frac{d}{dx} (\ln K_1) \frac{d\theta}{dx} + \left(\frac{2\pi}{\lambda_s} \right)^2 \frac{k_2}{k_1} \theta = 0 \quad (4)$$

(where $\ln K_1$ is the natural logarithm of K_1 , sometimes written as $\log_e K_1$). This may be reduced to nondimensional (or normalized) form by substituting

$$X = x/l_T \quad (5a)$$

and

$$\Omega_T = \frac{2\pi l_T}{\lambda_s} \left(\frac{k_2}{k_1} \right)^{1/2} = 2\pi \frac{l_T}{\lambda_T} \quad (5b)$$

where l_T is the resonant length of torsional vibration of the rod to be considered and λ_T is the wavelength of the torsional waves in an untapered rod of the same cross-sectional shape. Then

$$\theta''(X) + \theta'(X) \frac{d}{dX} (\ln K_1) + \Omega_T^2 \theta = 0 \quad (6)$$

where

$$\theta' = \frac{d\theta}{dx} \text{ and } \theta'' = \frac{d^2\theta}{dx^2}.$$

This may be compared with the corresponding nondimensional (or normalized) equation for longitudinal waves (see Equation 6 of my above-mentioned copending application),

$$U''(X) + U'(X) \frac{d}{dX} (\ln A) + \Omega_L^2 U = 0 \quad (7)$$

($X=x/l_L$, where l_L is the longitudinally resonant length, $U=u/l_L$, where $u(x)$ is the longitudinal displacement at x , $A(X)$ is the cross-sectional area, and $\Omega_L=2\pi l_L/\lambda$, where λ is the wavelength of extensional waves of the frequency considered in a uniform rod of the material).

The forms of the above Equations 6 and 7 are obviously the same, and it follows that if functions $U(\Omega_L, X)$, $A(\Omega_L, X)$ that together satisfy Equation 7 are found, as taught in my above-mentioned copending application, then functions $\theta(\Omega_T, X)$, $K_1(\Omega_T, X)$ of the same forms will satisfy Equation 6.

It can be shown that, as taught in my copending application, if it is wished to design a longitudinally resonant body having given performance characteristics, it is convenient to find a function $U(X)$ that satisfies the boundary conditions imposed, and then to determine the corresponding shape function, $A(X)$, from Equation 7.

The same approach may clearly be adopted in the torsional case. If due attention is paid to limits, Equation 6 may be inverted to give

$$\frac{d}{dX} (\ln K_1) = -\frac{\theta''(X) + \Omega_T^2 \theta}{\theta'} \quad (8)$$

The torsional problem may differ somewhat from the longitudinal one in that, while Ω_L , Equation 7, is a constant, Ω_T will be a function of X unless the cross-sectional contour or shape is of constant type (that is, circular, square, or the like) along the length of the body. If the cross-sectional contour or shape varied in a predetermined way along the length of the rod, then $\Omega_T(X)$ would be known explicitly and $\ln K_1$ could be found by integration of Equation 8, given $\theta(X)$ and its derivatives, as readily as if Ω_T were constant. The integral would, of course, depend on the arbitrary form of Ω_T . The most likely variation of cross-sectional contour or shape would be one in which one dimension were held constant, and all the

variation of section-size appeared in the other dimension. (One obvious example would be that of rectangular section with one side of the rectangle constant in length.) In that case Ω_T would be a function of $D(X)$ and therefore of $K_1(X)$, so that the integration of Equation 8 would involve appreciably more complexity.

As an arbitrary variation of cross-sectional contour or shape is a rather unlikely requirement to be encountered in practical applications, it will not be exhaustively analyzed at this time. It is felt that the principles of the invention will be more readily perceived if their application is described in connection with rods whose cross-sectional contours or shapes do not vary along their lengths, that is, rods for which Ω_T is constant.

Torsional amplitude transformers

For the purposes of the present application, an "amplitude transformer" is defined as a mechanically resonant body having two strain-free surfaces on one of which the amplitude of vibration is much bigger than on the other. In my above-mentioned copending application, I have used the semi-inverse approach outlined above to design a class of longitudinally vibrating amplitude transformers, designated as "Fourier" type, capable of giving large magnification without being unduly slender. It can be seen as demonstrated above that differential equations of the same form govern the longitudinal and torsional designs. Now consider the necessary boundary conditions which, as will presently become apparent, are not identical for the two designs.

The principal difference between the longitudinal and the torsional cases lies in the dependence of the strain (here the shear strain), S , as given by the relation

$$S = k_3 D(X) \frac{d\theta}{dx} = \frac{k_3}{l_T} D(X) \theta'(X) \quad (9)$$

where k_3 is another constant dependent on cross-sectional shape only. The engineering definition of shear strain is used, so that $k_3=1$ for a rod of circular cross section if $D(X)$ is the radius.

Unlike the strain in the longitudinal case (which is $U'(X)$) this strain depends on $D(X)$. However, since $D(X)$ is finite, the boundary condition " $S=0$ at a free surface" leads to $\theta'=0$, just as it leads to $U'=0$ in the longitudinal case. Using the same arguments as were used in my above-mentioned copending application, boundary conditions are imposed on $\theta(X)$ of the same form as those used for $U(X)$:

$$\theta(0) = \theta_0 \quad (10a)$$

$$\theta(1) = -M_0 \theta_0 \quad (10b)$$

$$\theta'(0) = 0 \quad (10c)$$

$$\theta'(1) = 0 \quad (10d)$$

$$\theta''(0) = -\Omega_T^2 \theta_0 \quad (10e)$$

$$\theta''(1) = M_0 \Omega_T^2 \theta_0 \quad (10f)$$

$$\theta'''(0) = 0 \quad (10g)$$

$$\theta'''(1) = 0 \quad (10h)$$

M_0 is the magnification required in angular displacement and velocity. Since the governing equation and the boundary conditions for the torsional and longitudinal amplitude transformers are of the same forms, displacement functions, $\theta(X)$, $U(X)$ of the same form will solve the problem. Using simpler notation than in my above-mentioned copending application,

$$\frac{\theta(X)}{\theta_0} = \frac{1}{2} (M_0 - 1) \sum_{j=0}^4 \beta_j \cos j\pi X \quad (11)$$

where

$$\beta_0 = 3\beta_4 - 4(4 - \gamma_T^2) \quad (12a)$$

$$\beta_1 = 2\mu(9 - \gamma_T^2) \quad (12b)$$

$$\beta_2 = -4(\beta_4 + \gamma_T^2) \quad (12c)$$

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$$\beta_3 = 2\mu(\gamma_T^2 - 1) \quad (12d)$$

$$\mu = (M_0 + 1)/(M_0 - 1) \quad (12e)$$

$$\gamma_T = \Omega_T/\pi = 2l_T/\lambda_T \quad (12f)$$

When the values for $\theta(X)$ and the consequent values of $\theta'(X)$ and $\theta''(X)$ are inserted into Equation 8 the following relation is obtained:

$$\frac{d(\ln K_1)}{d(\cos \pi X)} = \frac{(4 - \gamma_T^2)(\gamma_T^2 + \beta_4) - \mu(9 - \gamma_T^2)(\gamma_T^2 - 1) \cos \pi - (16 - \gamma_T^2)\beta_4 \cos^2 \pi X}{\mu(3 - \gamma_T^2) - 2(\gamma_T^2 + 2\beta_4) \cos \pi X + 3\mu(\gamma_T^2 - 1) \cos^2 \pi X + 4\beta_4 \cos^3 \pi X} \quad (12g)$$

For given values of M_0 , γ_T and β_4 , Equation 8 can now be integrated; in fact, the results of the integration in my above-mentioned copending application can be used, since, if

$$M_0 = M \quad (13)$$

$$\text{and } \gamma_T = \gamma$$

$$\text{and } \beta_4 = \alpha_{42}$$

where M , γ and α_{42} are the free parameters in the longitudinal case, then

$$\left[\frac{K_1(M_0, \gamma_T, \beta_4, X)}{K_1(M_0, \gamma_T, \beta_4, 0)} \right]_{\text{torsional}} = \left[\frac{A(M, \gamma, \alpha_{42}, X)}{A(M, \gamma, \alpha_{42}, 0)} \right]_{\text{longitudinal}} \quad (14a)$$

and since K_1 is proportional to D^4 and A is proportional to D^2 , subject to Conditions 13,

$$\left[\frac{D(X)}{D(0)} \right]_{\text{torsional}} = \left[\frac{D(X)}{D(0)} \right]_{\text{longitudinal}}^{1/2} \quad (14b)$$

Thus, the profiles of the Fourier torsional vibrators may be very simply deduced from those of the Fourier longitudinal vibrators with the corresponding free parameters. It remains to be seen whether the choice of these parameters made in my above-mentioned copending application is appropriate here.

Optimization

It has been shown that in any harmonically resonant body the maximum particle velocity, v_m , is limited by the maximum permissible strain. If the maximum strain is S_m and the speed of sound is c , then

$$v_m = \phi S_m c \quad (15)$$

ϕ is a function of geometry and of Poisson's ratio of the material. If S_m and c are suitably defined, ϕ can often be made independent of Poisson's ratio. Furthermore, if the intuitively obvious definitions of maximum strain and of speed of sound are used, then ϕ will be of the order of 1 for shapes one would ordinarily think of using as vibrators.

For one-dimensional torsional vibrators, the appropriate maximum strain is the maximum shear strain, Equation 9, the appropriate speed of sound is the speed of shear waves,

$$c_s = \left(\frac{n}{\rho} \right)^{1/2} = p\lambda_s/2\pi \quad (16)$$

The maximum particle velocity, v_m , occurs on the surface:

$$v_m = p k_4 |D(X)\theta(X)|_{\max} \quad (17)$$

where k_4 is again a function of cross-sectional shape only.

Therefore, from Equations 9 and 17

$$\phi = \frac{k_4}{k_3} \frac{p l_T}{c_s} \left| \frac{D(X)\theta(X)}{D(X)\theta'(X)} \right|_{\max} \quad (18a)$$

$$\phi = \frac{k_4}{k_3} \left(\frac{k_1}{k_2} \right)^{1/2} \Omega_T \left| \frac{D(X)\theta(X)}{D(X)\theta'(X)} \right|_{\max} \quad (18b)$$

It can readily be seen that the ratio of maxima on the right-hand side of Equation 18b is independent of the absolute cross-sectional size, as long as $D(X)/D(0)$ is unchanged. Since k_4/k_3 and k_4/k_2 are functions of cross-

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tional shape only and Ω_T is a chosen constant, ϕ is also independent of cross-sectional size. It follows, Equation 15, that the maximum particle velocity for given material and maximum strain is similarly independent. Therefore, any angular velocity, however large, can be obtained without overstraining by making the cross section small enough. This re-emphasizes the appropriateness of ϕ as

a criterion of the suitability of a shape for the production of high amplitude.

In the longitudinal case of my above-mentioned copending application the parameters γ and α_{42} were varied to provide shapes that were flexurally stiff and had high values of ϕ . An examination of the variation of ϕ and of flexural stiffness of the torsional designs with γ_T and β_4 shows that the same respective values of the latter parameters provide good compromises. The torsional designs were therefore computed by means of Equation 14b.

Computations

In the longitudinal case there was only one displacement, the longitudinal, to consider, and corresponding to it one magnification, M . In the torsional case, however, there is not only the angular displacement, $\theta(X)$, with its magnification, M_θ , to consider, but also it is likely to be at least as interesting to consider the surface displacement, $u_s(X)$, and its magnification, M_s . Mathematically, M_θ is equivalent to M , and is an input parameter of the problem. On the other hand,

$$M_s = M_\theta \frac{D(1)}{D(0)} \quad (19)$$

and cannot therefore be found until the profile has been determined.

There are two other quantities that will be tabulated that differ in definition from their longitudinal equivalents, namely ϕ and the impedance.

It can be seen from Equation 18b that ϕ is a function of cross-sectional shape. However, the quantity $(k_4/k_3) \cdot (k_1/k_2)^{1/2}$ does not differ greatly from 1 for shapes not having sharply re-entrant corners (for which it would be $\ll 1$), and it can only very slightly exceed 1 (1.03 for an equilateral triangle). The choice of cross-sectional shape is arbitrary, and so one may tabulate ϕ_0 , the value of ϕ for circular cross-section, for which

$$(k_4/k_3) \cdot (k_1/k_2)^{1/2} = 1$$

The impedance, $J_T(X)$, is defined as the ratio of the couple to the angular velocity at X . It should be noted that this has different dimensions from the impedance in the longitudinal case. It can be shown that

$$J_T(X) = \frac{i}{\pi} (k_1 k_2 n_p)^{1/2} \cdot D^4(0) \cdot P_T(X) \quad (20a)$$

where

$$P_T(X) = \frac{1}{\gamma_T} \left[\frac{D(X)}{D(0)} \right]^4 \frac{\theta'(X)}{\theta(X)} \quad (20b)$$

It follows that a rigid body of polar moment of inertia I_p may be matched to the vibrator where

$$|J_T(X)| = p I_p \quad (20c)$$

the body being substituted on the side of increasing X where $P_T(X)$ is positive and conversely.

From Equation 14b it can be seen that the torsional vibrator with a given value of M_0 is much less slender than the longitudinal vibrator with $M = M_0$, with the same length and the same maximum "diameter." For instance, if

$$R_D = \frac{D_{\max}}{D_{\min}} \quad (21)$$

then the longitudinal vibrators with $M=1000$ has $R_D=54$, while the torsional vibrator with $M_0=1000$ has $R_D=7.4$. It therefore becomes feasible to make and use torsional vibrators with very large values of M_0 . Unfortunately, as M_0 increases both the numerator and denominator of the right-hand side of Equation 8, corresponding to Equation 6a of my above-mentioned copending application, which are sums and differences of quantities of the order of 1, become very small. With $M_0 \geq 1000$, the normal eight-figure accuracy of the IBM 7090 computer becomes inadequate, and it becomes necessary to use double precision arithmetic. At present this takes the computer about ten times as long as does single precision arithmetic. The convergence of certain iterations in the programme becomes slower the larger M_0 , so that the computation for large values of M_0 becomes relatively expensive. For these reasons the largest value of M_0 that has here been computed is 30,000, although the value of $R_D(=21)$ is still quite modest. If specific needs arise for members having even much larger values of M_0 , the computations can, of course, readily be done.

Results

Table I, hereinbelow, gives the values of the chosen parameters, M_0 , β_4 , γ_T , and of the calculated quantities, M_s , ϕ and the position of the node. The illustrative cases tabulated range from $M_0=5$ to $M_0=30,000$ ($M_s=2.7$ to $M_s=1400$). The β_4 values in each case lie in the range -0.28 to -0.35 which is considered a useful range for the purposes described herein. Table II, hereinbelow, gives data for the determination of the profiles [as $D(X)/d(0)$], the displacements [as $U(X)/U(0)$] and the impedances [as $P_T(X)$], as functions of position along the member.

The positions are all given as distances from the low-amplitude end and expressed in terms of λ_T , where

$$\lambda_T = \lambda_s \left(\frac{k_1}{k_2} \right)^{1/2} \quad (22)$$

In an untapered rod of the same material and cross-sectional shape as the amplitude transformer torsional waves of frequency $p/2\pi$ will propagate with wavelength λ_T .

Fourth order Fourier torsional amplitude transformers symbols used in Tables I and II

MTheta=Magnification of angular displacement
MSurf=Magnification of surface-particle displacement
Beta4=Free parameter (Equation 11)
LambdaT=Wavelength of torsional waves (Equation 22)
Gamma= $2 \times \text{length} / \text{lambdaT}$
Phi=Figure of merit for ability to withstand large amplitude (Equations 15 and 18)
Z=(Distance from low-amplitude end)/lambdaT
ZNode=Z at displacement node
D/Do=Ratio of characteristic dimension of cross section at Z to that at Z=0
TH/TH0=Ratio of amplitude of angular displacement at Z to that at Z=0
SU/SU0=Ratio of amplitude of surface-particle displacement at Z to that at Z=0
PT=Reduced impedance at Z (Equation 20b)
Powers of 10 are denoted by E. For example, $1.23E+02$ and $-1.23E-02$ mean, respectively, 123. and -0.0123 .

TABLE I.—PARAMETERS AND DERIVED QUANTITIES

MTheta	MSurf	Beta4	Gamma	Phi	ZNode
5.00	2.713	-0.3000	1.29524	1.44366	0.24710
50.00	15.050	-0.3000	1.43224	1.40079	0.23898
100.00	25.116	-0.3400	1.46275	1.39541	0.23839
300.00	56.653	-0.3400	1.46875	1.39239	0.22948
700.00	105.121	-0.3350	1.46757	1.39182	0.21967
30000.00	1414.969	-0.3000	1.44744	1.39354	0.15878

TABLE II.—PROFILES, DISPLACEMENTS AND IMPEDANCES
[MTheta=5.00]

Z	D/DO	TH/TH0	SU/SU0	PT
0.	1.00000	1.00	1.00	0.
0.03238	0.99991	0.98	0.98	-0.648E+00
0.06476	0.99979	0.92	0.92	-0.135E+01
0.09714	0.99988	0.82	0.82	-0.220E+01
0.12952	1.00014	0.69	0.69	-0.332E+01
0.16190	0.99965	0.53	0.53	-0.508E+01
0.19429	0.99604	0.34	0.34	-0.862E+01
0.22667	0.98525	0.14	0.14	-0.224E+02
0.25905	0.96256	-0.09	-0.08	0.366E+02
0.29143	0.92509	-0.33	-0.31	0.900E+01
0.32381	0.87449	-0.62	-0.54	0.452E+01
0.35619	0.81670	-0.96	-0.79	0.266E+01
0.38857	0.75865	-1.37	-1.04	0.166E+01
0.42095	0.70543	-1.85	-1.31	0.107E+01
0.45333	0.65963	-2.40	-1.58	0.703E+00
0.48572	0.62198	-3.00	-1.86	0.467E+00
0.51810	0.59230	-3.60	-2.13	0.310E+00
0.55048	0.57004	-4.15	-2.37	0.201E+00
0.58286	0.55460	-4.60	-2.55	0.121E+00
0.61524	0.54554	-4.90	-2.67	0.566E-01
0.64762	0.54255	-5.00	-2.71	0.
[MTheta=100.00]				
0.	1.00000	1.00	1.00	0.
0.03657	0.99732	0.97	0.97	-0.732E+00
0.07314	0.99262	0.90	0.89	-0.154E+01
0.10971	0.99254	0.77	0.76	-0.255E+01
0.14627	0.99870	0.60	0.60	-0.406E+01
0.18284	0.99416	0.41	0.41	-0.687E+01
0.21941	0.93705	0.17	0.16	-0.179E+02
0.25598	0.81528	-0.21	-0.17	0.146E+02
0.29255	0.68008	-0.92	-0.63	0.315E+01
0.32912	0.56822	-2.34	-1.33	0.117E+01
0.36569	0.48382	-4.96	-2.40	0.516E+00
0.40226	0.42108	-9.36	-3.94	0.251E+00
0.43883	0.37412	-16.05	-6.00	0.133E+00
0.47540	0.33861	-25.32	-8.57	0.750E-01
0.51196	0.31161	-37.06	-11.55	0.446E-01
0.54853	0.29111	-50.66	-14.75	0.275E-01
0.58510	0.27574	-65.02	-17.93	0.173E-01
0.62167	0.26458	-78.65	-20.81	0.108E-01
0.65824	0.25700	-89.93	-23.11	0.636E-02
0.69481	0.25260	-97.39	-24.60	0.295E-02
0.73138	0.25116	-100.00	-25.12	0.
[MTheta=700.00]				
0.	1.00000	1.00	1.00	0.
0.03669	0.98380	0.97	0.96	-0.721E+00
0.07338	0.95798	0.89	0.85	-0.147E+01
0.11007	0.96035	0.75	0.72	-0.230E+01
0.14676	0.99871	0.59	0.59	-0.381E+01
0.18345	0.94853	0.40	0.38	-0.644E+01
0.22014	0.72929	-0.01	-0.01	0.306E+03
0.25683	0.54288	-1.35	-0.73	0.196E+01
0.29351	0.42291	-5.17	-2.19	0.492E+00
0.33020	0.34437	-14.08	-4.85	0.171E+00
0.36689	0.29043	-31.67	-9.20	0.713E-01
0.40358	0.25189	-62.02	-15.62	0.336E-01
0.44027	0.22353	-108.76	-24.31	0.174E-01
0.47696	0.20226	-173.85	-35.16	0.974E-02
0.51365	0.18615	-256.47	-47.74	0.576E-02
0.55034	0.17394	-352.30	-61.28	0.354E-02
0.58703	0.16479	-453.50	-74.73	0.223E-02
0.62372	0.15816	-549.58	-86.92	0.139E-02
0.66041	0.15365	-629.08	-96.66	0.816E-03
0.69710	0.15103	-681.62	-102.95	0.378E-03
0.73379	0.15017	-700.00	-105.12	0.
[MTheta=30000.00]				
0.	1.00000	1.00	1.00	0.
0.03619	0.81514	0.96	0.79	-0.549E+00
0.07237	0.69730	0.80	0.56	-0.903E+00
0.10856	0.88245	0.61	0.54	-0.144E+01
0.14474	0.61838	0.45	0.28	-0.264E+01
0.18093	0.32610	-2.61	-0.85	0.453E+00
0.21711	0.21797	-22.03	-4.80	0.524E-01
0.25330	0.16168	-93.66	-15.14	0.119E-01
0.28949	0.12757	-288.42	-36.79	0.369E-02
0.32567	0.10505	-719.46	-75.58	0.140E-02
0.36186	0.08934	-1536.91	-137.30	0.611E-03
0.39804	0.07794	-2905.84	-226.49	0.298E-03
0.43423	0.06947	-4967.86	-345.11	0.158E-03
0.47042	0.06306	-7793.34	-491.44	0.898E-04
0.50660	0.05817	-11336.68	-659.49	0.538E-04
0.54279	0.05445	-15408.83	-839.04	0.334E-04
0.57897	0.05165	-19679.49	-1016.55	0.211E-04
0.61516	0.04962	-23713.84	-1176.67	0.133E-04
0.65135	0.04823	-27040.01	-1304.26	0.781E-05
0.68753	0.04743	-29233.63	-1386.54	0.362E-05
0.72372	0.04717	-30000.00	-1414.97	0.263E-12

It can be seen from Table I that for all values of M_0 , ϕ_0 is substantially 1.4, while in the longitudinal vibrators of my above-mentioned copending application, ϕ is sub-

stantially 1.6. Thus the torsional transformers are slightly less efficient producers of large particle velocity for given maximum strain than are the longitudinal transformers. They are, however, still much better than a "stepped" torsional transformer, consisting of two quarter-wave cylinders of substantially differing diameters having a common longitudinal axis and joined by a fillet, would be. Such a "stepped" transformer would have, as discussed in my copending application for the longitudinal case, a value of $\varphi \leq 0.8$.

Although the value of φ is only slightly smaller for the torsional than for the comparable longitudinal transformer, care must be taken in comparing the maximum particle velocities of which the two types of vibrator are capable. This velocity is equal to φS_{mc} , Equation 15. The speed of sound, c , is smaller for shear waves than for extensional waves in the ratio $1:[2(1+\nu)]^{1/2}$ where ν is Poisson's ratio. If the material is liable to failure by a plastic process such as yield or fatigue, as discussed by applicant in an article published in the periodical "Nature," volume 188, pages 1183 and 1184, London, December 31, 1960, then the maximum shear stress is a good criterion of the limit at which one may expect failure. The maximum (engineering) shear strain is then related to the tensile strain in plane stress as $(1+\nu):1$. Therefore, for such materials:

$$\frac{(v_m/\varphi)_{\text{torsional}}}{(v_m/\varphi)_{\text{extensional}}} = \left(\frac{1+\nu}{2}\right)^{1/2} = 0.81 \text{ for } \nu=0.3 \quad (23a)$$

If, on the other hand, the material is essentially brittle, so that the maximum principal tensile strain is the criterion of the limit at which one may expect failure, then the maximum permissible (engineering) shear strain in pure shear is twice the maximum permissible tensile strain in plane stress and therefore

$$\frac{(v_m/\varphi)_{\text{torsional}}}{(v_m/\varphi)_{\text{extensional}}} = \left(\frac{2}{1+\nu}\right)^{1/2} = 1.24 \text{ for } \nu=0.3 \quad (23b)$$

"Engineering" shear strain is discussed, for example, in the above-mentioned book by W. P. Mason, entitled "Physical Acoustics and the Properties of Solids," at page 356. Similarly, elastic fracture and flow is discussed, for example, in a book of J. C. Jaeger directed to this subject, published by John Wiley & Sons, New York, second edition, 1962.

Thus, a Fourier torsional amplitude transformer with $\varphi=1.4$, if made of a ductile metal with $\nu=0.3$, is capable of 30 percent smaller maximum particle velocity than is the corresponding longitudinal transformer with $\varphi=1.6$. But if it is made of a brittle material with $\nu=0.3$, the torsional transformer is capable of 10 percent greater particle velocity than the longitudinal. In assessing this it should be remembered that, because λ_s is substantially 0.6λ , the torsional transformer is correspondingly smaller than the comparable longitudinal transformer. This is illustrated in FIG. 2 of the accompanying drawing where members 14 and 18 represent the size and contour of comparable torsional and longitudinal transformers, respectively.

A further important object of the invention is, as mentioned above, to produce transformers of high magnification that are not too slender. In FIGS. 3 and 4 the performance in this respect is shown by characteristic 20 of FIG. 3 for angular magnification, M_θ , and for magnification of surface-particle velocity, M_s , by characteristic

former for angular magnification. FIG. 4, characteristic 32, illustrates the relative performance of either a torsionally resonant or longitudinally resonant exponentially tapered transformer for surface-particle velocity or linear magnification. It can be shown that an exponential transformer has $M_\theta=R_D^2$ and $M_s=R_D$. It is, however, clear, from the characteristics as above noted, that, except at very small magnifications, the torsional Fourier transformers are very much less slender, and consequently much stiffer, than the comparison (exponential) type. This is emphasized in FIGS. 1 and 2, as noted hereinabove, where profiles or longitudinal contours of five transformers of magnification 700 are drawn. It is clear from the stubbiness of the Fourier transformer with $M_\theta=700$, characteristic 10 of FIG. 1, that transformers of this type having extremely large angular magnifications are perfectly practical to make and operate.

For surface magnification (FIGS. 4 and 2), where the exponential transformer becomes quite unmanageable at high magnification, since the larger portion of its length becomes impracticably slender, the torsional Fourier transformer has to compete with the Fourier longitudinal transformer. The latter has the advantage, from the point of view of slenderness, that the wavelength for longitudinal waves is greater than that for torsional waves, which allows larger diameters to be used for a given frequency.

Another criterion of slenderness is the normalized flexural compliance, C_0 , which is defined as the inverse of stiffness in my above-mentioned copending application. Allowance must, of course, be made for the difference in wavelength between torsion and extension. In FIG. 5, the flexural compliance, C_0 , is plotted against magnification, assuming Poisson's ratio to be 0.3. Characteristics 40, 42 and 44 of FIG. 5 are, as designated, those for the Fourier torsional angular magnification, Fourier torsional linear magnification, and Fourier longitudinal types of amplitude transformers, respectively. This figure shows not only how very stiff the transformers of angular velocity are, but also how much stiffer the torsional transformers of particle velocity are than their longitudinal counterparts, except at very small magnification. Thus, for example, the torsional transformer with $M_s=700$ is about 20 times stiffer than the longitudinal one with $M=700$.

Numerous and varied modifications and rearrangements of the above described specific embodiments may readily be devised by those skilled in the art without departing from the spirit and scope of the principles of the invention. The embodiments described are illustrative only and are not to be taken as limiting the invention.

What is claimed is:

1. A mechanical transformer for magnifying the amplitude of a repetitive, vibratory, torsional physical displacement by a factor M_θ , M_θ having a value exceeding 2, said transformer comprising an elongated acoustically resonant member of constant cross-sectional shape or contour having one end larger than the other end, a length l_T less than a wavelength λ_T for torsional waves of the member's resonant frequency propagated along a uniform member of the same cross-sectional shape and the same material, a maximum transverse dimension not greater than one-half of said wavelength, the torsional rigidity K_1 of the member varying between the larger and smaller ends of the member in accordance with the integral of the relation

$$\frac{d(\ln K_1)}{d(\cos \pi X)} = \frac{(4-\gamma_T^2)(\gamma_T^2+\beta_1)-\mu(9-\gamma_T^2)(\gamma_T^2-1)\cos \pi X - (16-\gamma_T^2)\beta_1 \cos^2 \pi X}{\mu(3-\gamma_T^2)-2(\gamma_T^2+2\beta_1)\cos \pi X + 3\mu(\gamma_T^2-1)\cos^2 \pi X + 4\beta_1 \cos^3 \pi X}$$

where

$$\gamma_T = \frac{2l_T}{\lambda_T}, \mu = \frac{M_\theta+1}{M_\theta-1}, X = \frac{x}{l_T}$$

30 of FIG. 4, respectively. For comparison, characteristic 34 of FIG. 4 is shown for the longitudinal Fourier transformer for linear magnification and characteristic 22 of FIG. 3 for the torsionally resonant exponential trans-

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where x represents the distance, along the member from its larger end, of the cross section for which the torsional rigidity K_1 is being instantly determined and β_4 is a constant at choice having a value between minus one and plus one.

2. The transformer of claim 1 in which γ_T is chosen to give substantially the lowest ration of maximum to minimum cross-sectional areas for the selected value of β_4 .

3. The transformer of claim 2 in which β_4 has a value between -0.28 and -0.35 .

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4. The transformer of claim 1 in which β_4 has a value between -0.28 and -0.35 .

5. The transformer of claim 1 in which β_4 is zero.

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