July 13, 1937. S. P. MEAD 2,086,629

SHIELDED CABLE SYSTEM

Filed April 14, 1936 2 Sheets-Sheet 1

FIG. 1

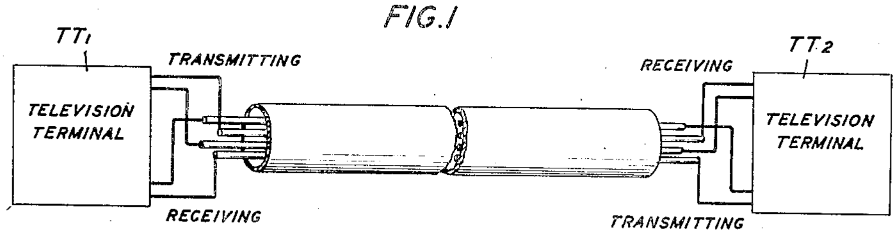


FIG. 3

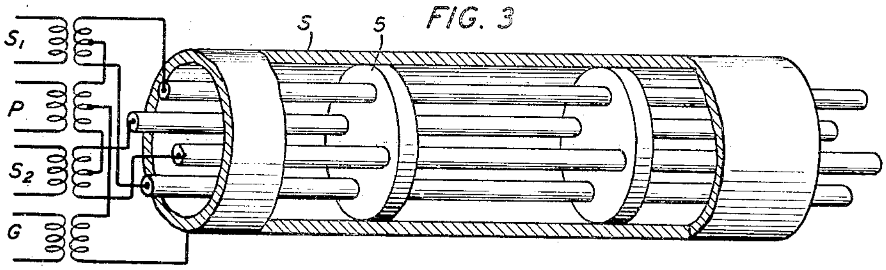


FIG. 4

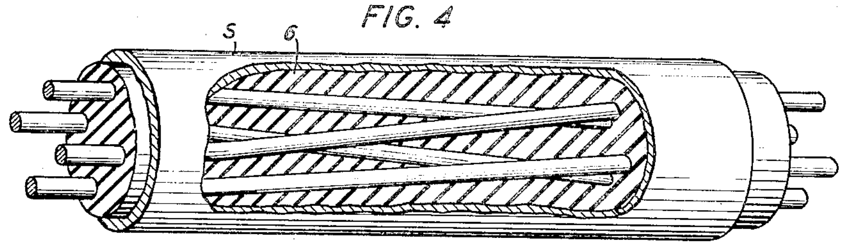


FIG. 7

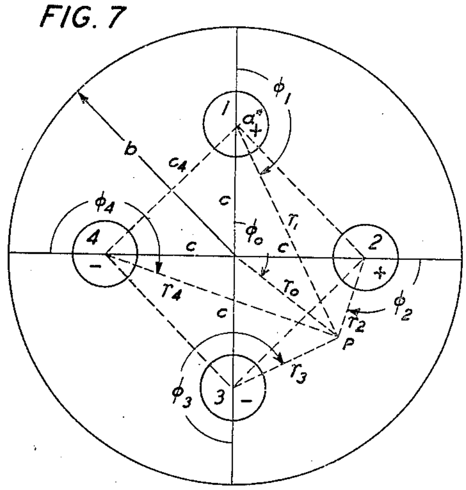
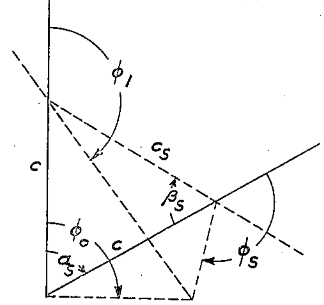


FIG. 7A



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2,086,629

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FIG. 2

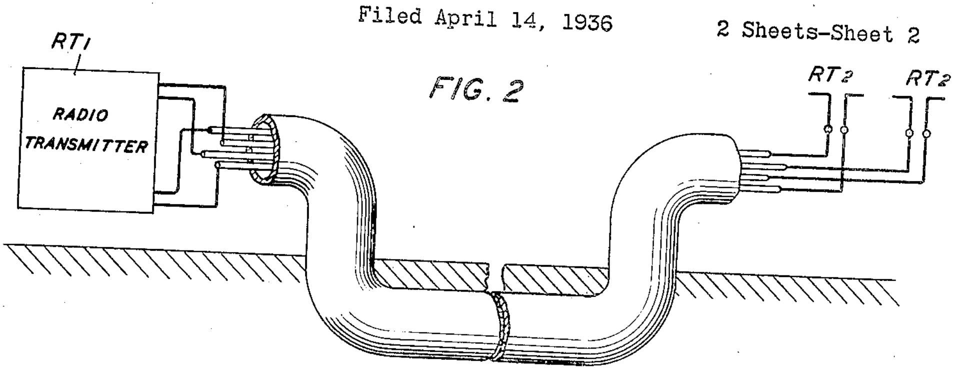


FIG. 6

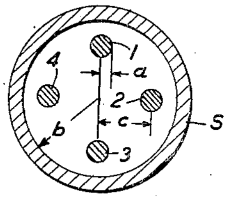


FIG. 5

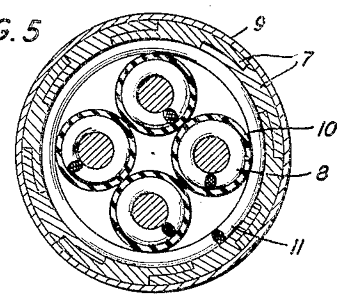


FIG. 9

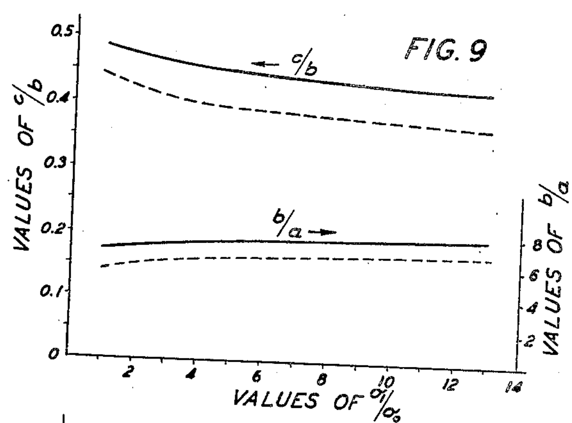


FIG. 10

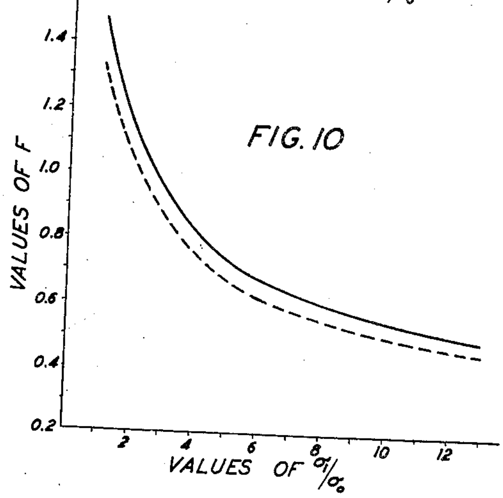
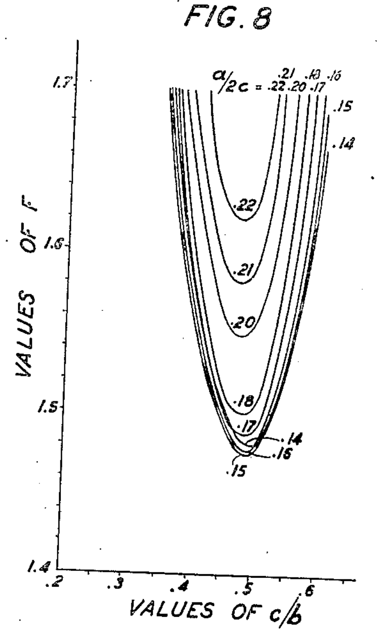


FIG. 8



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Patented July 13, 1937 2,086,629

UNITED STATES PATENT OFFICE

2,086,629

SHIELDED CABLE SYSTEM

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Application April 14, 1936, Serial No. 74,269

9 Claims. (Cl. 178—44)

This invention relates to electrical transmission systems and more particularly to high frequency signaling systems comprising several conductors disposed within a metallic shield.

For the transmission of high frequency signals such as employed in wide band carrier telephone and television systems shielded conducting systems comprising one or more pairs of conductors balanced with respect to ground have heretofore been proposed. Such shielded systems are characterized by low susceptibility to interference arising from external sources, this being due in part to the shield surrounding the conductors and in part to the electrical balance of the pairs with respect to ground. Conversely, and for the same reasons, waves transmitted through the shielded system have little or no disturbing effect on adjacent external signaling circuits. A further characteristic is, or may be, moderate attenuation even at frequencies of several hundred or several thousand kilocycles per second and higher. As with all familiar types of conducting systems, however, the attenuation increases as the frequency rises and this becomes a limiting factor in the design of the entire system.

A particular object of the present invention is to reduce the attenuation of signals in a shielded high frequency transmission system comprising two pairs of conductors. More specifically, the object is to provide a conducting system of this type in which the dimensions and spacing of the conductors are optimum with respect to signal attenuation.

The optimum design in accordance with the present invention is dependent on several variable factors. Applicant has discovered the critical relationships that must exist between the several variables and these are set forth in mathematical formulae that appear in a subsequent portion of this specification.

The nature of the present invention and other objects and features thereof will appear more fully in the detailed description that follows, reference being made to the accompanying drawings, in which:

Figs. 1 and 2 show typical four-wire transmission systems embodying the present invention;

Figs. 3, 4, and 5 show in partial section shielded two-pair cables in accordance with the invention;

Fig. 6 is a cross-sectional view of a typical shielded two-pair cable;

Figs. 7 and 7A are diagrams used in connection with the mathematical demonstration and description of applicant's invention;

Fig. 8 is a family of curves showing the relation between attenuation and the geometric proportioning of a shielded two-pair cable in which the conducting wires and the shield have the same conductivity;

Fig. 9 shows the relation, optimum with respect to low attenuation, between the dimensional ratios of a shielded two-pair cable and the ratio of the respective conductivities of the shield and the conducting wires; and

Fig. 10 is a graph showing the relation between minimum attenuation and the conductivity ratio in shielded two-pair cables of optimum configuration.

Referring now to the drawings, there is shown in Fig. 1 a typical transmission system in accordance with the invention comprising television terminals TT_1 and TT_2, connected by a transmission line comprising four conductors symmetrically disposed within a copper tube and substantially air insulated therefrom. Two diagonally opposite conductors comprise a circuit and of the two circuits shown one may be utilized for transmission in the east-west direction and the other for transmission in the west-east direction. Repeaters may be installed in the line at suitable intervals. In this embodiment it is presumed that both circuits are operated in the same frequency range and that this may be as much as a megacycle in width. Alternatively, transmission in two directions may be effected over each circuit utilizing different portions of the frequency range for separation of the oppositely directed signals.

Another typical embodiment of the present invention is represented in Fig. 2 where a radio transmitter RT_1 is connected to two independent dipole antennas RT_2 over a shielded two-pair cable similar to that represented in Fig. 1 and utilized in substantially the same manner for one-way transmission, diagonally opposite conductors again comprising each of the two circuits.

Figs. 3, 4, and 5 show cable structures suitable for use in transmission systems such as shown in Figs. 1 and 2. In Fig. 3 the four conductors are maintained in spaced relation with respect to the copper tube S by means of perforated insulating washers 5 spaced at intervals such that the dielectric between the conductors and the sheath is substantially gaseous. Fig. 3 shows also how four signaling circuits may be derived from the four conductors and sheaths. Two side circuits S_1 and S_2 are obtained simply by connection to each pair of diagonally opposite conductors. A phantom circuit P is derived by connections to the mid-point of the two side circuits, in this case each pair of diagonally opposite conductors being utilized as one side of the phantom circuit. A fourth, "ghost" circuit G is derived by means of a connection to the mid-point of the phantom circuit and a connection to the sheath, the four conductors in parallel being utilized as one side of the circuit. The four conductors may be laid parallel to the axis as shown in Fig. 3 or disposed with a slight helical twist as shown in Fig. 4.

The two-pair cable represented in Fig. 4 is characterized chiefly by the fact that a solid dielectric material 6 is utilized to maintain the four conductors in their relative positions with respect to each other and the sheath S.

The shielded two-pair cable illustrated in Fig. 5 comprises a sheath made up of a number of interengaging profiled strips 7 laid up with a slight helical twist and bound together by an external helical wrapping of iron or brass tape 9. Each of the four conductors has applied to it a helical cord 8 of any suitable insulating material and an over-all wrapping of paper 10. Another insulating cord 11 is wrapped around the four insulated conductors holding them in position with respect to the shield.

In any of the cables hereinbefore described the conductors may be of such type that currents of frequencies well above the audible range travel substantially on the outer surfaces of the conductors. They may be, for example, either solid or tubular. In the latter case the thickness of the walls of the tube may be small compared to the diameter. The conductors may alternatively consist of a cylindrical assembly of conducting strips, tapes, ribbons, uninsulated wires or the like. The latter form of construction might be particularly desirable if the conductors are large and if a flexible structure is required.

The four conductors comprising the cable may be either parallel or transposed at frequent intervals. In either case the conductors may be considered parallel for mathematical purposes and the optimum diameter and spacing ratios will be the same in either case. One method of transposing is to twist the conductors helically around the axis of the shield as illustrated.

The shield, in addition to performing an electrical function by protecting the circuit from external induction, may be useful in affording mechanical protection to the circuit and thereby permitting the use of an air dielectric to a considerable extent. As the high frequency current penetrates into the shield and conductors very little because of skin effect, the electrical requirements are satisfied by a thin shield and thin walled conductors. Consequently, the thickness of the shield and the thickness of the walls of the conductors will ordinarily be determined by mechanical considerations. The thickness of the shield will usually be such that it does not enter into the problem of determining the optimum configuration of conductors and shield.

The use of the shield will ordinarily make it possible where desired to allow the signals transmitted over the path to drop down to a minimum level determined by the noise due to thermal agitation in the conductors. Hence the use of a shield may allow the spacing of intermediate amplifiers at wider intervals in the circuits than would otherwise be possible.

Fig. 6 is a cross-section of a shielded two-pair cable, on which are indicated the principal variables: the radius a of the individual conductors, the internal radius b of the sheath, and the distance c from the axis of each conductor to the axis of the sheath. The last-mentioned dimension will hereinafter be sometimes referred to as the interaxial separation of the conductors and the sheath.

Our object is to determine the ratio of the inner diameter of the shield to the diameter of the conductors and the ratio of the diagonal interaxial separation of the conductors to the inner diameter of the shield that makes the high frequency attenuation a minimum for any given inner diameter of shield. This will be done by first developing a general expression for the attenuation of such a system as a function of these ratios and then determining the value of these ratios which will make the attenuation a minimum for any predetermined size of shield.

The attenuation is also a function of the conductivity of conductors and shield. Therefore, the values of the aforementioned ratios will be found for values of the ratio, σ_1/σ_0, of the conductivity of the wires to that of the shield, varying, for specific example, from 1 to 13. The ratio $\sigma_1/\sigma_0=1$, for example, may correspond to a system of copper conductors with copper shield, and the ratio $\sigma_1/\sigma_0=13$, to copper conductors with lead shield.

Thus, a formula will be derived for the attenuation of a wave of frequency above the audible range propagated in a pair of long straight parallel wires of circular cross-section, with a similar pair at right angles to the first and with the two pairs enclosed symmetrically in a hollow conducting shield of circular cross-section the axis of which is parallel to and equally distant from the axes of the four wires. The system, therefore, consists of two balanced pairs. While the attenuation depends only upon the dimensions and electrical constants of the system, nevertheless arbitrary currents I_1 and I_2 will be assumed in the two pairs. The current I_2 might, of course, be zero. That is, the second circuit may or may not be energized.

In the shielded four-wire system there are two circuits available in addition to the two-side circuits. One of these is the phantom of the four-wire circuit, in which, of course, the two wires of each of the two side circuit pairs are used in parallel to form the sides of the phantom circuit. The other available circuit is that formed by the four wires in multiple with the shield as the return, the latter being sometimes called the "ghost" or "wraith" circuit. These circuits correspond to the three possible modes of propagation in a system of balanced pairs; that is, side circuit, phantom circuit, and sheath (or ground) return. Each mode may exist alone, or theoretically, any two or all three may exist simultaneously without cross-talk from one another. (See "Propagation of periodic currents over a system of parallel wires" by John R. Carson and Ray S. Hoyt, Bell System Technical Journal, July, 1927.)

Formulae may be developed for the phantom and sheath-return circuits by mathematical analyses similar to that for the side circuit, and the proportioning determined to provide minimum attenuation for these circuits independently. Were all three of these circuits or any two of them to be used simultaneously the proportioning might be on the basis of the maximum total frequency range obtainable for a given size of shield and any given maximum allowable attenuations for the circuits used.

The shielded phantom circuit has an advantage over the other four-wire circuits and over the shielded pair on the score of the reduction which it affords in cross induction with neighboring circuits. On the other hand, in systems of the proportions of cable configurations, the attenuation in the phantom is about twice that in the pair for the same size of shield owing to the comparatively large ratio of the diameter of the wires to the spacing. In other words, in order to obtain the same attenuation for the phantom as for the pair or side circuit, the system must be twice as large as for the latter.

As for the sheath-return circuit, while the attenuation is of the same order of magnitude as in the side circuit, the induction with neighboring circuits would be large. This circuit might, however, have special uses such as signaling.

In the schematic cross-section of the configuration shown in Fig. 7, conductors No. 1 and No. 3 form the first pair and No. 2 and No. 4, the second pair. The notation for coordinate systems and dimensions will be clear from Fig. 7. For convenience, the shield is assumed of infinite extent, its thickness being of no significance in the present problem. The conductors are assumed nonmagnetic and the leakage conductance is assumed to be zero.

It is well known that the attenuation, α, per unit length of a conducting system, when the leakage is zero and the frequency is so high that ω^2L^2 is large compared with R^2, is given by

$$\alpha=\frac{R}{2}\sqrt{\frac{C}{L}} \qquad (1)$$

where R, L, and C are the effective resistance, inductance and capacity of the system per unit length. Denoting by Z the impedance of the system per unit length, we have

$$Z=R+i\omega L=2Z_1+4i\omega\log(2c/a)+\Delta Z, \qquad (2)$$

where Z_1 is the internal impedance of the wire per unit length with concentric return and ΔZ the impedance increment or proximity effect correction due to the presence of the shield and second pair of wires and the reaction of the wires of the first pair upon each other. The inductance L may be written

$$L=L_i+L_e$$

where L_i is the internal inductance of wires and sheath and L_e the external inductance of the system. By external inductance is meant that portion of the inductance which is independent of the conductivity of the conductors and which may therefore be determined by assuming infinite conductivity. Hence, we have

$$L_eC=\mu k$$

where μ and k are, respectively, the permeability and specific inductive capacity of the dielectric in electromagnetic c. g. s. units. Thus, putting $\mu=1$,

$$C=k/L_e \qquad (3)$$

Hence, when the impedance Z has been determined, the attenuation readily follows from Equations (1), (2), and (3). The immediate object then is to derive the formula for the impedance Z.

With the notation,

$\lambda=a/2c$,
$\epsilon=c/b$,
a=radius of wires in cms.,
b=radius of sheath in cms.,
c=interaxial separation of wires and sheath in cms.,
k=specific inductive capacity of dielectric in electrostatic c. g. s. units,
=l for air,
v=velocity of light=3×10^{10} cms. per second,
σ_0=conductivity of sheath in electromagnetic c. g. s. units,
σ_1=conductivity of wires in electromagnetic c. g. s. units,
Z_1=internal impedance of wire with concentric return,
Z_0=internal impedance of sheath with concentric return,

the impedance Z may be written,

$$\begin{aligned}Z&=R+i\omega L\\&=2Z_1F_1+8Z_0F_0+i\omega L_e\end{aligned} \qquad (4)$$

where

$$F_0=\frac{\epsilon^2}{1-\epsilon^4}-\delta_0-\delta_0'$$

$$F_1=1+\delta_1+\delta_1'$$

$$L_e=4\log\frac{1}{\lambda}\frac{1-\epsilon^2}{1+\epsilon^2}-\delta-\delta'$$

$$\begin{aligned}\delta_0=&\frac{2\lambda^2\epsilon^2}{1-\lambda^2}\left(1+\frac{1-8\epsilon^2+\epsilon^4}{(1-\epsilon^4)^2}\right)+\frac{6\lambda^2\epsilon^6}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)\\&-\frac{8\lambda^2\epsilon^8}{1-\lambda^2}\frac{1+\epsilon^4}{(1-\epsilon^4)^2}+\frac{16\lambda^4\epsilon^4}{1-\lambda^2}\frac{1+\epsilon^4}{(1-\epsilon^4)^2}(3-8\epsilon^2)\\&-\frac{4\lambda^4\epsilon^2}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(3-16\epsilon^2)\end{aligned}$$

$$\begin{aligned}\delta_1=&\frac{2\lambda^2}{(1-\lambda^2)^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(1-4\epsilon^2)\\&+2\lambda^4(1+16\epsilon^6)-\frac{8\lambda^2\epsilon^6}{(1-\lambda^2)^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)\\&+16\lambda^4\epsilon^2\frac{2-\lambda^2}{(1-\lambda^2)^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(3-8\epsilon^2)\end{aligned}$$

$$\begin{aligned}\delta=&\frac{4\lambda^2}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(1-4\epsilon^2)+2\lambda^4(1+16\epsilon^6)\\&-\frac{16\lambda^2\epsilon^6}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)+\frac{32\lambda^4\epsilon^2}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(3-8\epsilon^2)\end{aligned}$$

$$\begin{aligned}\delta_0'=&\frac{4\lambda^2\epsilon^2}{1+\lambda^2}\left[1-2\frac{\epsilon^2}{1+\epsilon^4}+(1-3\epsilon^4+5\epsilon^8)(1-2\epsilon^2)\right]\\&-\frac{12\lambda^2\epsilon^6}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)+\frac{8\lambda^2\epsilon^8}{1+\lambda^2}(1-3\epsilon^4+5\epsilon^8)\\&+\frac{16\lambda^4\epsilon^2}{1+\lambda^2}(1-3\epsilon^4+5\epsilon^8)(3-8\epsilon^2)\\&-\frac{8\lambda^4\epsilon^2}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)(3-16\epsilon^2)\end{aligned}$$

$$\begin{aligned}\delta_1'=&\frac{8\lambda^2}{(1+\lambda^2)^2}(1-2\epsilon^2)\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)\\&+32\lambda^4(1-4\epsilon^6)+\frac{16\lambda^2\epsilon^6}{(1+\lambda^2)^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)\\&+32\lambda^4\epsilon^2\frac{2+\lambda^2}{(1+\lambda^2)^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)(3-8\epsilon^2)\end{aligned}$$

$$\begin{aligned}\delta'=&\frac{4\lambda^2}{1+\lambda^2}(1-2\epsilon^2)\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)+32\lambda^4(1-4\epsilon^6)\\&+\frac{32\lambda^2\epsilon^6}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)+\frac{64\lambda^4\epsilon^2}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)(3-8\epsilon^2)\end{aligned}$$

Upon the assumption of circular symmetry in the current distribution in the wires (equivalent to the concentration of current on the axes), the δ's in Formula (4) are all zero; that is,

$$Z=2Z_1+8Z_0\frac{\epsilon^2}{1-\epsilon^4}+4i\omega\log\frac{1}{\lambda}\frac{1-\epsilon^2}{1+\epsilon^2} \qquad (5)$$

or

$$\Delta Z=\Delta^{(o)}Z$$

$\Delta^{(0)}Z$ representing the impedance increment due to the loss in the shield. Thus Formula (5) gives the total impedance when the wires are very thin (filamentary) or when they are composed of insulated strands, and it is the same with or without the second pair.

When the wires are solid, tubular or composed of uninsulated strands, δ_1', δ_0', and δ_{12}' represent the effect of the presence of the second pair of wires. Without the second pair of wires Z is given by putting $\delta_1'=\delta_0'=\delta_{12}'=0$.

When the conductors are solid, tubular or composed of uninsulated strands, however, we may ignore the internal inductance, L_i, of the conductors as compared to the external inductance L_e of the system. In that case, that is, when $L_i/L_e<<1$, Equation (1) may be written

$$\alpha=\frac{\sqrt{k}}{v}R/2L_e \qquad (6)$$

On this basis the required attenuation is given by

$$\alpha=\frac{\sqrt{k}}{v}R_0F \qquad (I)$$

$$=\frac{\sqrt{k}}{v}R_1F' \qquad (II)$$

where

$$R_0=\frac{1}{b}\sqrt{\frac{f}{\sigma_0}}=$$

high frequency resistance of shield

$$R_1=\frac{1}{a}\sqrt{\frac{f}{\sigma_1}}=$$

high frequency resistance of wire

$$F=\frac{\frac{1}{2\epsilon\lambda}\sqrt{\frac{\sigma_0}{\sigma_1}}F_1+4F_0}{L_e}$$

$$F'=\frac{F_1+8\epsilon\lambda\sqrt{\sigma_1/\sigma_0}F_0}{L_e}$$

For two pairs without the sheath, obtained by putting $b=\infty$ or $\epsilon=0$, in (I), we have the formula

$$\alpha=\frac{\sqrt{k}}{v}R_1F' \qquad (III)$$

where

$$F'=\frac{F_1}{L_e}$$

and

$$F_1=1+\frac{2\lambda^2}{(1-\lambda^2)^2}+\frac{8\lambda^2}{(1+\lambda^2)^2}+34\lambda^4$$

$$L_e=4\log\frac{1}{\lambda}-\frac{4\lambda^2}{1-\lambda^2}-\frac{4\lambda^2}{1+\lambda^2}-34\lambda^4$$

The formula for a single pair shielded is obtained from Equation (I) simply by putting $\delta_0'=\delta_1'=\delta_{12}'=0$ and without the shield, by then putting $\epsilon=0$.

I proceed to the derivation of the impedance Z from the geometry and electrical constants of the system. The method is as follows (referring to the cross-sectional diagram of Fig. 7).

The axial electric force E_z and the tangential magnetic force H_θ (which will be represented by E and H, respectively) at any point in the dielectric are given in terms of a vector potential A and a scalar potential V by the relations,

$$E=-\text{grad } V-i\omega A$$

$$H=\text{curl } A \qquad (7)$$

Putting $F=i\omega A$ and $\gamma=-\partial/\partial z$, we have

$$E=\gamma V-F \qquad (8)$$

which must hold at the surfaces of the conductors.

Representing by K and γ the characteristic impedance and propagation constant per unit length, by $V_1^{(1)}$ and $V_1^{(3)}$ the potential at $r_1=a$ and $r_3=a$, respectively, and by I_1 the current in wire No. 1, we have

$$K=\frac{V^{(1)}-V^{(3)}}{I_1}$$

$$\gamma K=R+i\omega L=\frac{\gamma V^{(1)}-\gamma V^{(3)}}{I_1} \qquad (9)$$

Thus we have

$$R+i\omega L=\frac{E^{(1)}+F^{(1)}}{I_1}-\frac{E^{(3)}+F^{(3)}}{I_1} \qquad (10)$$

the Superscripts (1) and (3) indicating the values of E and F at the surfaces $r_1=a$ and $r_3=a$, respectively.

We must now express E and F in terms of I_1. Inside the conductors, E must satisfy the wave equation. It may, therefore, be written

$$E=Z_1I_j\frac{J_0(r_jz_1/a)}{J_0(z_1)}+\sum_{n=1}^{\infty}J_n(r_jz_1/a)[g_{nj}\cos n\phi_j+g'_{nj}\sin n\phi_j], \qquad (11)$$

$$0\leqq r_j\leqq a$$

inside conductor No. j and

$$E=\sum_{n=1}^{\infty}K_n(r_0z_0/b)[h_n\cos n\phi_0+h_n'\sin n\phi_0], \qquad (12)$$

$$b\leqq r_0\leqq\infty,$$

inside the sheath where g_{nj}, g_{nj}', h_n and h_n' are arbitrary constants and

Z_1=internal impedance of wire with concentric return,

$J_n(z)$ and $K_n(z)$=Bessel functions of first and second kinds of order n and complex argument z,

$$z_1=ia\sqrt{4\pi\sigma_1\mu_1 i\omega}$$

$$z_0=ib\sqrt{4\pi\sigma_0\mu_0 i\omega}$$

μ_1 and μ_0=permeability of wire and sheath, respectively,

σ_1 and σ_0=conductivity of wire and sheath, respectively,

$$\frac{\omega}{2\pi}=\text{frequency}$$

$$i=\sqrt{-1}$$

The wave function F satisfies Laplace's equation in two dimensions and may be expressed as the sum of four convergent waves centered on the axes of each of the four wires, respectively, and a reflected wave centered on the axis of the sheath. Thus, F may be written

$$F=F_0+F_1+F_2+F_3+F_4 \qquad (13)$$

where

$$F_0=-\sum_{n=1}^{\infty}r_0^n(A_{n0}\cos n\phi_0+B_{n0}\sin n\phi_0),$$

$$F_j=-2\mu i\omega A_{0j}\log r_j+\sum_{n=1}^{\infty}\frac{A_{nj}\cos n\phi_j+B_{nj}\sin n\phi_j}{r_j^n},$$

and

$$j=1, 2, 3, 4$$

A_{n0}, A_{nj}, B_{n0}, and B_{nj} are arbitrary constants to be determined by boundary conditions. We may immediately reduce the number of arbitrary constants by considering that the field at any point (r_0,ϕ_0) is the negative of the field at the point $(r_0,=\phi_0+\pi)$ or that

$$F(r_0,\phi_0)=-F(r_0,\phi_0+\pi)$$

We have, therefore, on revising the constants to avoid double subscripts,

$$F_0=\sum_{n=1}^{\infty} r_o^{\,n}(W_n \cos n\phi_0+W_n' \sin n\phi_0) \qquad (14)$$

$$F_1=-2\mu i\omega I_1 \log r_1+\sum_{n=1}^{\infty}\frac{A_n \cos n\phi_1+B_n \sin n\phi_1}{r_1^{\,n}} \qquad (15)$$

$$F_2=-2\mu i\omega I_2 \log r_2+\sum_{n=1}^{\infty}\frac{C_n \cos n\phi_2-D_n \sin n\phi_2}{r_2^{\,n}} \qquad (16)$$

$$F_3=2\mu i\omega I_1 \log r_3-\sum_{n=1}^{\infty}\frac{A_n \cos n\phi_3+B_n \sin n\phi_3}{r_3^{\,n}} \qquad (17)$$

$$F_4=2\mu i\omega I_2 \log r_4-\sum_{n=1}^{\infty}\frac{C_n \cos n\phi_4-D_n \sin n\phi_4}{r_4^{\,n}} \qquad (18)$$

The boundary conditions at $r_1=a$ and $r_2=a$ are

$$\frac{\partial F}{\partial r_i}=-\frac{\mu\partial E}{\mu_1\partial r_i} \qquad (19)$$

and

$$\frac{\partial F}{\partial \phi_i}=-\frac{\partial E}{\partial \phi_i}\; j=1,\,2 \qquad (20)$$

and at $r_0=b$,

$$\frac{\partial F}{\partial r_0}=-\frac{\mu\partial E}{\mu_0\partial r_0} \qquad (21)$$

and

$$\frac{\partial F}{\partial \phi_0}=-\frac{\partial E}{\partial \phi_0} \qquad (22)$$

To apply these boundary conditions, E and F must be transformed to the coordinate systems r_1,ϕ_1; r_2,ϕ_2; and r_0,ϕ_0. Performing this transformation, as we shall show below, gives for F in the neighborhood of $r_1=a$,

$$\begin{aligned}F=&-2\mu_1 i\omega I_1 \log \frac{r_1}{2c}\\ &-\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n A_n-2\sum_{n=1}^{\infty}\sin\frac{n\pi}{4}\left(-\frac{1}{c_2}\right)^n D_n\\ &+\sum_{n=1}^{\infty}\left(c^{2n-1}\right)W_{2n-1}+\sum_{n=1}^{\infty}\cos n\phi_1\left[\frac{A_n}{r_1^{\,n}}+G_n r_1^{\,n}\right]\\ &+\sum_{n=1}^{\infty}\sin n\phi_1\left[\frac{B_n}{r_1^{\,n}}+H_n r_1^{\,n}\right] \qquad (23)\end{aligned}$$

where G_n and H_n are expressible in I_1, I_2 and the arbitrary constants A_n, B_n, C_n, D_n, W_n and W_n'. A similar expression is obtainable for F at $r_2=a$ with, we may say, G_n' and H_n' corresponding to G_n and H_n.

In the neighborhood of $r_0=b$, F is given by

$$\begin{aligned}F=&-\sum_{n=1}^{\infty} r_0^{\,2n-1}[W_{2n-1}\cos(2n-1)\phi_0+\\ &W'_{2n-1}\sin(2n-1)\phi_0]+\sum_{n=1}^{\infty}\left(\frac{c}{r_0}\right)^n N_n \cos n\phi_0+\\ &\sum_{n=1}^{\infty}\left(\frac{c}{r_0}\right)^n O_n \sin n\phi_0 \qquad (24)\end{aligned}$$

where O_n and N_n are expressible in terms of A_n, B_n, C_n, D_n, I_1 and I_2.

Introducing Equations (11), (12), (23), and (24) in the boundary relations (19)–(22) and equating harmonic coefficients will give 12 simultaneous equations to determine the arbitrary constants A_n, B_n, C_n, D_n, W_n, W_n' and the six constants pertaining to the interior of the wires and sheath.

Now turning back to Equation (10), it is apparent from the boundary relations that the harmonic terms in the expressions $E^{(1)}+F^{(1)}$ vanish, leaving only the fundamental. From symmetry it is also evident, then, that

$$E^{(3)}+F^{(3)}=-(E^{(1)}+F^{(1)})$$

We thus obtain (with $I_1=1$),

$$R+i\omega L=2(E^{(1)}+F^{(1)})=2Z_1+4i\omega\log\frac{2c}{a}+\Delta Z \qquad (25)$$

where

$$\begin{aligned}\Delta Z=&-2\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n A_n-\\ &4\sum_{n=1}^{\infty}\left(-\frac{1}{c_2}\right)^n D_n \sin\frac{n\pi}{4}+\\ &2\sum_{n=1}^{\infty} c^{2n-1}W_{2n-1}\end{aligned}$$

This completes the formal solution of the problem. However, as a straightforward solution of the doubly infinite set of equations for the arbitrary constants is not possible, the method of approximation which has been applied to analogous problems in wave propagation over parallel conductors to obtain the correction due to proximity effect, will be given below. We shall first add the details which were omitted in the outline of the formal solution.

The following relations among the coordinate systems are used to express G_n, H_n, G_n', H_n', O_n, and N_n in the arbitrary constants A_n, B_n, C_n, D_n, W_n, and W_n'. These are derived in Note II of the paper "Transmission Characteristics of the Submarine Cable", by Carson and Gilbert, Jour. Franklin Institute, Dec., 1921. Refering to Fig. 7A, we have

$$\begin{aligned}\frac{\cos n\phi_s}{r_s^{\,n}}=&\frac{(-1)^n}{c_s^{\,n}}\Big[\cos n\beta_s-\frac{nr_1}{1!c_s}\cos(\phi_1-(n-1)\beta_s)+\\ &\frac{n(n+1)}{2!}\left(\frac{r_1}{c_s}\right)^2\cos(2\phi_1-(n-2)\beta_s)-\\ &\frac{n(n+1)(n+2)}{3!}\left(\frac{r_1}{c_s}\right)^3\cos(3\phi_1-(n-3)\beta_s)+\dots\Big]\\ &s=2,\,3,\,4 \qquad (26)\end{aligned}$$

$$\begin{aligned}\frac{\sin n\phi_s}{r_s^{\,n}}=&\frac{(-1)^n}{c_s^{\,n}}\Big[\sin n\beta_s+\frac{nr_1}{1!c_s}\sin(\phi_1-(n-1)\beta_s)-\\ &\frac{n(n+1)}{2!}\left(\frac{r_1}{c_s}\right)^2\sin(2\phi_1-(n-2)\beta_s)+\\ &\frac{n(n+1)(n+2)}{3!}\left(\frac{r_1}{c_s}\right)^3\sin(3\phi_1-(n-3)\beta_s)-\dots\Big] \qquad (27)\end{aligned}$$

$$\begin{aligned}\log r_s=&\log c_s+\\ &\frac{r_1}{c_s}\cos(\phi_1+\beta_s)-\frac{1}{2}\left(\frac{r_1}{c_s}\right)^2\cos 2(\phi_1+\beta_s)+\\ &\frac{1}{3}\left(\frac{r_1}{c_s}\right)^2\cos 3(\phi_1+\beta_s)-\dots \qquad (28)\end{aligned}$$

$$\begin{aligned}r_0^{\,n}\cos n\phi_0=&\\ &c^n\Big(1+\frac{n}{1!}\left(\frac{r_1}{c}\right)\cos\phi_1+\frac{n(n-1)}{2!}\left(\frac{r_1}{c}\right)^2\cos 2\phi_1+\\ &\frac{n(n-1)(n-2)}{3!}\left(\frac{r_1}{c}\right)^3\cos 3\phi_1+\dots\Big) \qquad (29)\end{aligned}$$

$$\begin{aligned}r_0^{\,n}\sin n\phi_0=&c^n\Big[\frac{n}{1!}\left(\frac{r_1}{c}\right)\sin\phi_1+\\ &\frac{n(n-1)}{2!}\left(\frac{r_1}{c}\right)^2\sin 2\phi_1+\dots\Big] \qquad (30)\end{aligned}$$

$$\begin{aligned}\log r_0=&\log c+\frac{r_1}{c}\cos\phi_1-\\ &\frac{1}{2}\left(\frac{r_1}{c}\right)^2\cos 2\phi_1+\frac{1}{3}\left(\frac{r_1}{c}\right)^3\cos 3\phi_1-\dots \qquad (31)\end{aligned}$$

$$\begin{aligned}\frac{\cos n\phi_s}{r_s^{\,n}}=&\frac{1}{r_0^{\,n}}\Big[\cos n(\phi_0-\alpha_s)+\\ &n\frac{c}{r_0}\cos(n+1)(\phi_0-\alpha_s)+\\ &\frac{n(n+1)}{2!}\left(\frac{c}{r_0}\right)^2\cos(n+2)(\phi_0-\alpha_s)+\dots\Big] \qquad (32)\end{aligned}$$

$$\frac{\sin n\phi_s}{r_s^n}=\frac{1}{r_0^n}\Big[\sin n(\phi_0-\alpha_s)+ n\frac{c}{r_0}\sin(n+1)(\phi_0-\alpha_s)+ \frac{n(n+1)}{2!}\left(\frac{c}{r_0}\right)^2\sin(n+2)(\phi_0-\alpha_s)+\ldots\Big] \quad (33)$$

$$P_n'=c^nW_n'+(n+1)c^{n+1}W_{n+1}'+\frac{(n+1)(n+2)}{2!}c^{n+2}W_{n+2}'+\ldots$$

$$W_{2n}=W_2'n=0$$

Similarly, in the neighborhood of $r_2=a$, F may be written,

$$F=-2\mu i\omega I_2\log\frac{r_2}{2c}-\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^nC_n-2\sum_{n=1}^{\infty}\left(-\frac{1}{c_2}\right)^n\sin\frac{n\pi}{4}B_n-\sum_{n=1}^{\infty}(-1)^nc^{2n-1}W_2'{}_{n-1}+$$
$$\sum_{n=1}^{\infty}\cos n\phi_2\left[\frac{C_n}{r_2^n}-\left(-\frac{r_2}{2c}\right)^n\left(\frac{2\mu i\omega I_2}{n}-S_C\right)+2\left(-\frac{r_2}{c_2}\right)^nT_B-\left(\frac{r_2}{c}\right)^nQ_n'\right]+$$
$$\sum_{n=1}^{\infty}\sin n\phi_2\left[-\frac{D_n}{r_2^n}+\left(-\frac{r_2}{2c}\right)^nS_D+2\left(-\frac{r_2}{c_2}\right)^nT_A+\frac{4\mu i\omega I_1}{n}\left(-\frac{r_2}{c_2}\right)^n\sin\frac{n\pi}{4}+\left(\frac{r_2}{c}\right)^nQ_n\right] \quad (35a)$$

where

$$Q_n=(-1)^n\left[c^nW_n+(n+1)c^{n+1}W_{n+1}-\frac{(n+1)(n+2)}{2!}c^{n+2}W_{n+2}-\frac{(n+1)(n+2)(n+3)}{3!}c^{n+3}W_{n+3}+\ldots\right]$$

$$\log r_s=\log r_0-\frac{c}{r_0}\cos(\phi_0-\alpha_s)-\frac{1}{2}\left(\frac{c}{r_0}\right)^2\cos 2(\phi_0-\alpha_s)-\ldots \quad (34)$$

$$Q_n'=(-1)^n[c^nW_n'+(n+1)c^{n+1}W_{n+1}'-\ldots]$$

and S_C, S_D, T_A, and T_B will be clear from the similar expressions in Equation (35).

Applying relations (32)-(34), F, in the neighborhood of $r_0=b$ is given by,

$$F=\sum_{n=1}^{\infty}r_0^{2n-1}\Big[W_{2n-1}\cos(2n-1)\phi_0+W_2'{}_{n-1}\sin(2n-1)\phi_0\Big]+$$
$$\sum_{n=1}^{\infty}\left(\frac{c}{r_0}\right)^{2n-1}\cos(2n-1)\phi_0\left[\frac{4i\omega I_1}{2n-1}+2M_A-(-1)^n2M_D\right]-$$
$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{c}{r_0}\right)^{2n-1}\sin(2n-1)\phi_0\left[\frac{4i\omega I_2}{2n-1}+2M_C-(-1)^n2M_B\right] \quad (36)$$

In the present problem,

$$\alpha_s=(s-1)\frac{\pi}{2}$$

or

$$\alpha_1=0,\ \alpha_2=\frac{\pi}{2},\ \alpha_3=\pi,\ \alpha_4=\frac{3\pi}{2}$$

and

$$\beta_s=\frac{\pi}{2}-\frac{\alpha_s}{2}=\frac{\pi}{4}(3-s)$$

or

$$\beta_2=\frac{\pi}{4},\ \beta_3=0,\ \beta_4=-\frac{\pi}{4}$$

Applying Formulas (26)-(31), F, in the neighborhood or $r_1=a$, becomes

where

$$M_A=\frac{A_1}{c}+\frac{2n-}{1!}\frac{2A_2}{c^2}+\frac{(2n-2)(2n-3)}{2!}\frac{A_3}{c^3}+\ldots+\frac{A_{2n-1}}{c^{2n-1}},$$

$$M_B=\frac{B_1}{c}+\frac{2n-2B_2}{1!\ c^2}+\frac{(2n-2)(2n-3)B_3}{2!\ c^3}+\ldots+\frac{B_{2n-1}}{c^{2n-1}},\ \ldots \quad (36a)$$

We are now in a position to introduce E and F in the boundary relations (19)-(22) to obtain the arbitrary constants. Putting $\mu=\mu_0=\mu_1=1$, equating harmonic coefficients and solving, gives

$$F=-2\mu_1 i\omega I_1\log\frac{r_1}{2c}-\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^nA_n+$$
$$\sum_{n=1}^{\infty}c^{2n-1}W_{2n-1}-2\sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{4}\right)\left(-\frac{1}{c_2}\right)^nD_n+\sum_{n=1}^{\infty}(\cos n\phi_1)\left[\frac{A_n}{r_1^n}-\left(-\frac{r_1}{2c}\right)^n\left(\frac{2\mu_1 i\omega I_1}{n}-S_A\right)+\right.$$
$$\left.\left(\frac{r_1}{c}\right)^nP_n+2\left(-\frac{r_1}{c_2}\right)^nT_D\right]+\sum_{n=1}^{\infty}(\sin n\phi_1)\left[\frac{B_n}{r_1^n}-2\left(-\frac{r_1}{c_2}\right)^n\left(\frac{2\mu_1 i\omega I_2}{n}\sin\frac{n\pi}{4}+T_C\right)+\left(\frac{r_1}{c}\right)^nP_n'-\left(-\frac{r_1}{2c}\right)^nS_B\right] \quad (35)$$

where

$$S_A=\frac{A_1}{2c}-(n+1)\frac{A_2}{(2c)^2}+\frac{(n+1)(n+2)}{2!}\frac{A_3}{(2c)^3}+\ldots$$

$$S_B=\frac{B_1}{2c}-(n+1)\frac{B_2}{(2c)^2}+\frac{(n+1)(n+2)}{2!}\frac{B_3}{(2c)^3}+\ldots$$

$$T_C=\frac{C_1}{c_2}\sin(1-n)\frac{\pi}{4}-(n+1)\frac{C_2}{c_2^2}\sin(2-n)\frac{\pi}{4}+\frac{(n+1)(n+2)C_3}{2!\ c_2^3}\sin(3-n)\frac{\pi}{4}+\ldots$$

$$T_D=\frac{D_1}{c_2}\sin(1-n)\frac{\pi}{4}-(n+1)\frac{D_2}{c_2^2}\sin(2-n)\frac{\pi}{4}+\frac{(n+1)(n+2)D_3}{2!\ c_2^3}\sin(3-n)\frac{\pi}{4}+\ldots$$

$$P_n=c^nW_n+(n+1)c^{n+1}W_{n+1}+\frac{(n+1)(n+2)}{2!}c^{n+2}W_{n+2}+\ldots$$

the equations (remembering that $\lambda=a/2c$ and $\epsilon=c/b$),

$$\frac{A_n}{(2c)^n}=\lambda^n\tau_n\left[(-\lambda)^n\left(\frac{2i\omega I_1}{n}-S_A\right)-\left(\frac{a}{c}\right)^nP_n-2\left(-\frac{a}{c_2}\right)^nT_D\right] \quad (37)$$

$$\frac{D_n}{(2c)^n}=\lambda^n\tau_n\left[2\left(-\frac{a}{c_2}\right)^n\left(\frac{2i\omega I_1\sin n\pi/4}{n}+T_A\right)+\left(\frac{a}{c}\right)^nQ_n+(-\lambda)^nS_D\right] \quad (38)$$

$$-c^{2n-1}W_{2n-1}=\epsilon^{4n-2}\eta_{2n-1}\left[\frac{4i\omega I_1}{2n-1}+2M_A-(-1)^n2M_D\right] \quad (39)$$

which are required to obtain the constants in ΔZ, Equation (25). τ_n and η_n are given by

$$\tau_n=\frac{z_1 J_n'-nJ_n}{z_1 J_n'+nJ_n}$$

$$=1-\frac{2n}{z_1}\frac{J_n}{J_{n-1}}$$

$$\longrightarrow\ 1-i\frac{2n}{z_1}\text{ when } z_1\ \longrightarrow\ \infty$$

$$\eta_n=\frac{z_0 K_n'+nK_n}{z_0 K_n'-nK_n}$$

$$=1-\frac{2n}{z_0}\frac{K_n}{K_{n+1}}$$

$$\longrightarrow\ 1-i\frac{2n}{z_0}\text{ when } z_0\ \longrightarrow\ \infty$$

(When the arguments of the Bessel functions are omittted z_1 is to be understood with J_n and z_0 with K_n.) The following method of successive approximations is found applicable to the solution of Equations (37)-(39).

We represent ΔZ by the sum

$$\Delta Z=\Delta^{(0)}Z+\Delta^{(1)}Z+\Delta^{(2)}Z+\ .\ .\ .$$

where the increments $\Delta^{(0)}Z$, $\Delta^{(1)}Z$, . . . are obtained by the following procedure.

1. Neglecting the summations in A_n and D_n in Equation (39), to determine a first approximation $W^{(0)}{}_{2n-1}$ to W_{2n-1}, we have, for $I_1=1$,

$$-c^{2n-1}W_{2n-1}=\frac{4i\omega}{2n-1}\eta_{2n-1}\epsilon^{4n-2}$$

Since

$$Z_0=-\frac{2i\omega K_0}{z_0 K_0'}\ \longrightarrow\ -\frac{2\omega}{z_0}\quad\text{when } z_0\ \longrightarrow\ \infty$$

we may write

$$\eta_{2n-1}=1+i(2n-1)\frac{Z_0}{\omega}$$

and

$$\Delta^{(0)}Z=-2\sum_{n=1}^{\infty}c^{2n-1}W^{(0)}{}_{2n-1}=$$

$$8Z_0\frac{\epsilon^2}{1-\epsilon^4}+4i\omega\log\frac{1-\epsilon^2}{1+\epsilon^2}\tag{40}$$

This impedance increment is exactly the same as that obtained when the second pair is absent, as is to be expected since the ignoring of the series in A_n and D_n is equivalent to the assumption of uniform current distribution over the surfaces of the wires (i. e., circular symmetry about the axes). Thus $\Delta^{(0)}Z$ represents the impedance increment if the wires are filamentary or if they are composed of insulated strands.

2. Proceeding to the boundary relations at $r_1=a$ and $r_2=a$, and putting

$$S_A=\frac{A_1^{(1)}}{2c}$$

$$T_D=0$$

$$P_n=cW_1^{(0)}+3c^3W_3^{(0)}+\ .\ .\ .$$
$$=4i\omega\epsilon^2/(1-\epsilon^4)-4Z_0\epsilon^2(1+3\epsilon^4+5\epsilon^8+\ .\ .\ .)$$

and

$$S_d=\frac{D_1^{(1)}}{2c}$$

$$T_A=0$$

$$Q_n=-cW_1^{(0)}+3c^3W_3^{(0)}-5c^5W_5^{(0)}+\ .\ .\ .$$
$$=4i\omega\epsilon^2/(1+\epsilon^4)-4Z_0\epsilon^2(1-3\epsilon^4+5\epsilon^8-\ .\ .\ .)$$

gives

$$\frac{A_1^{(1)}}{2c}=\frac{\lambda^2\tau_1}{1-\lambda^2\tau_1}(2P_n-2i\omega)$$

and

$$\frac{D_1^{(1)}}{2c}=\frac{\lambda^2\tau_1}{1+\lambda^2\tau_1}(2Q_n-4i\omega)$$

But we have

$$Z_1=\frac{2i\omega J_0}{z_1 J_0'}\ \longrightarrow\ -\frac{2\omega}{z_1},\text{ when } z_1\ \longrightarrow\ \infty$$

Simplifying, we get

$$\frac{A_1^{(1)}}{2c}=-2i\omega\frac{\lambda^2}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)+Z_1\frac{2\lambda^2}{(1-\lambda^2)^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)-Z_0\frac{8\epsilon^2\lambda^2}{1-\lambda^2}(1+3\epsilon^4+5\epsilon^8+.\ .\ .),\tag{41}$$

$$\frac{D_1^{(1)}}{2c}=-4i\omega\frac{\lambda^2}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)+Z_1\frac{4\lambda^2}{(1+\lambda^2)^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)-Z_0\frac{8\epsilon^2\lambda^2}{1+\lambda^2}(1-3\epsilon^4+5\epsilon^8-.\ .\ .)\tag{42}$$

3. Now returning to the surface of the shield, we include the A_n and D_n series in boundary Equation (39). Then, putting

$$M_A=2\frac{A_1^{(1)}}{2c}$$

and

$$M_D=2\frac{D_1^{(1)}}{2c}$$

to obtain $W_1^{(1)}$ in terms of $A_1^{(1)}$ and $D_1^{(1)}$, we have

$$-cW_1^{(1)}=4\eta_1\epsilon^2\left(\frac{A_1^{(1)}}{2c}+\frac{D_1^{(1)}}{2c}\right)$$

Now we may approximate ΔZ more closely by adding the increment,

$$\Delta^{(1)}Z=\frac{2A_1^{(1)}}{2c}+\frac{4D_1^{(1)}}{2c}-2cW_1^{(1)}\tag{43}$$

and writing

$$\Delta Z=\Delta^{(0)}Z+\Delta^{(1)}Z\tag{44}$$

$$=2Z_1\left[\frac{2\lambda^2}{(1-\lambda^2)^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(1-4\epsilon^2)+\frac{8\lambda^2}{(1+\lambda^2)^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)(1-2\epsilon^2)\right]$$
$$+8Z_0\left[\frac{\epsilon^2}{1-\epsilon^4}-\frac{2\lambda^2\epsilon^2}{1-\lambda^2}\left(1+\frac{1-8\epsilon^2+\epsilon^4}{(1-\epsilon^4)^2}-\frac{4\lambda^2\epsilon^2}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right.\right.\right.$$
$$\left.\left.+(1-2\epsilon^2)(1-3\epsilon^4+5\epsilon^8-\ .\ .\ .\ )\right]$$
$$+4i\omega\left[\log\frac{1-\epsilon^2}{1+\epsilon^2}-\frac{\lambda^2}{1-\lambda^2}\left(1-\frac{4\epsilon^2}{1-\epsilon^4}\right)(1-4\epsilon^2)-\frac{4\lambda^2}{1+\lambda^2}\left(1-\frac{2\epsilon^2}{1+\epsilon^4}\right)(1-2\epsilon^2)\right]$$

We find, however, that the terms $A_2^{(1)}/(2c)^2$, $D_2^{(1)}/(2c)^2$, $c^3W_3^{(1)}$ may be appreciable. These are obtained from Equations (37)-(39) by putting $n=2$. Thus, a more exact value of $\Delta^{(1)}Z$ is given by

$$\Delta^{(1)}Z=2\frac{A_1^{(1)}}{2c}+4\frac{D_1^{(1)}}{2c}+2cW_1^{(1)}-2\frac{A_2^{(1)}}{(2c)^2}-8\frac{D_2^{(1)}}{(2c)^2}+2c^3W_3^{(1)}\tag{44}$$

4. Proceeding to the next increment in ΔZ we have

$$\Delta^{(2)}Z=2\frac{A_1^{(2)}}{(2c)^2}+4\frac{B_1^{(2)}}{(2c)^2}+2cW_1^{(2)}\tag{45}$$

This degree of approximation includes all terms containing λ^4. Terms containing higher powers of λ may be ignored.

Thus we have

$$\Delta Z = \Delta^{(0)}Z + \Delta^{(1)}Z + \Delta^{(2)}Z$$

or

$$Z = 2Z_1F_1 + 8Z_0F_0 + i\omega L_e \qquad (46)$$

where F_0, F_1 and L_e are given by Equation (4). Finally,

$$R = 2R_1F_1 + 8R_0F_0 \qquad (47)$$

and

$$L = 2L_1F_1 + 8L_0F_0 + L_e = L_i + L_e \qquad (48)$$

We have

$$Z_0 = R_0 + i\omega L_0 = \frac{1}{b}\sqrt{\frac{f}{\sigma_0}}(1+i) \qquad (49)$$

and

$$Z_1 = R_1 + i\omega L_1 = \frac{1}{a}\sqrt{\frac{f}{\sigma_1}}(1+i) \qquad (50)$$

Introducing R and L_e in Equation (3) gives α as in (I). Since

$$F' = \frac{a}{b}\sqrt{\frac{\sigma_1}{\sigma_0}}F$$

Equation (II) is a direct consequence of Equation (I).

Owing to the complexity of the expression, the most practical way to determine the values of ϵ and λ which make the attenuation α a minimum, b being assumed a constant, is to substitute various pairs of values of b/a and c/b in the expression F and determine graphically the pair which makes F a minimum. Fig. 8 shows a family of attenuation curves with the parameter $a/2c$ and abscissae c/b for the case of a conductivity ratio σ_1/σ_0, of unity. To obtain the attenuation, α, in decibels per mile, the ordinates, F, are to be multiplied by the factor

$$\frac{0.0466}{b}\sqrt{\frac{kf}{\sigma_0}} = \frac{1.96}{b}\sqrt{\frac{\sigma_1}{\sigma_0}}\sqrt{kf}$$

for copper wires with $\sigma_1 = 0.565 \times 10^{-3}$,

where

f = frequency in megacycles,
b = radius of shield in centimeters,
k = specific inductive capacity of dielectric in electrostatic c. g. s. units (=1 for air),
σ_0 = conductivity of shield in electromagnetic c. g. s. units,
σ_1 = conductivity of wires in electromagnetic c. g. s. units.

The optimum dimensional ratios are thus seen to be $c/b=0.49$ and $a/2c=0.15$, giving $b/a=6.8$. Fig. 9 shows the optimum values of c/b and b/a for values of σ_1/σ_0 from 1 to 13, the dotted curves representing the corresponding optimum dimensional ratios for a single pair cable. These are the conditions for minimum attenuation when the inner diameter of the shield is fixed. It may be observed that the optimum proportioning ratios are independent of the frequency, dielectric constant and absolute size of conductors.

Fig. 8 shows the minimum value of attenuation in decibels per mile, when $\sigma_1/\sigma_0=1$, to be

$$(0.0466/b)\ (\sqrt{kf/\sigma_0})(1.475)$$

or for a system of copper conductors,

$$(1.96\sqrt{kf}/b)(1.475)$$

The solid curve of Fig. 10 represents the minimum attenuation for the two pair shielded cable for values of σ_1/σ_0 from 1 to 13. To obtain decibels per mile, multiply the ordinates by the factor

$$\frac{0.0466}{b}\sqrt{\frac{kf}{\sigma_0}} = \frac{1.96}{b}\sqrt{\frac{\sigma_1}{\sigma_0}}\sqrt{kf}$$

for copper wires. The dotted curve represents the minimum attenuation in a single pair shielded cable. It is obvious that the attenuation of the system can be reduced by increasing the size of the shield, keeping the ratios b/a and c/b fixed. The diameter of the shield would probably be determined by such considerations as the maximum frequency to be transmitted over the system and the maximum allowable attenuation at that frequency. It will be noted that the attenuation in the two pair shielded cable may be made equal to that in the single pair shielded cable by increasing the radius of the shield about 10 per cent.

It is interesting to note that the fixed ratios give fixed values of inductance, capacity and characteristic impedance regardless of the actual size of the system. Thus, by substituting the ratios b/a and c/b as given by Fig. 9 into Expressions (3) and (4) and changing to practical units, the inductance and capacity can be obtained. They are, respectively, $L=0.733$ milhenry per mile and $C=0.0393$ microfarad per mile when $\sigma_1/\sigma_0=1$ and $k=1$. The value of the high frequency characteristic impedance for the optimum configuration becomes

$$K = \sqrt{L/C} = 137 \text{ ohms}$$

Obviously, the data for the solution of the converse problem of the optimum size and proportioning of a cable to have a predetermined attenuation is also provided by Figs. 9 and 10. Thus, to design a two-pair cable of conductivity σ_1 with a shield of conductivity σ_0 to have a given attenuation α, introduce the appropriate F from Fig. 10 in Equation (I) to obtain b (in centimeters) corresponding to the given value of α (in decibels per mile). Then determine a and c from the ratios c/b and b/a in Fig. 9. It is interesting to note that the ratio a/c is practically constant with respect to the conductivity ratio.

For example, suppose we require the optimum dimensions for an attenuation of 6 decibels in a system of copper wires within a copper sheath for a frequency of one million cycles per second when $k=1.07$. We then have

$$\frac{\sigma_1}{\sigma_0}=1,\ \sigma_0=0.565\times10^{-3}$$

$$f=1,\ \alpha=6,\ k=1.07.$$

From Fig. 10, $F=1.475$.

From Equation (I),

$$b=\frac{0.0466}{\alpha}\sqrt{\frac{kf}{\sigma_0}}F$$

$$=0.50 \text{ cm.}=0.197 \text{ in.}$$

From Fig. 9,

$$c/b=0.49 \text{ and } b/a=6.8$$

Thus,

$$c=0.0965 \text{ in. and } 2a=0.058 \text{ in.}$$

In the determination of the configuration for minimum attenuation it was assumed that the value of leakage conductance was zero. However, in practice it will be necessary to support the two conductors and to keep them in the desired relative position with respect to the shield.

This will require the use of a certain amount of dielectric material inside the shield which will consequently introduce a certain amount of leakage conductance, and increase the average dielectric constant of the system.

By making such supports out of materials having low loss and low dielectric constant and by spacing them as far apart as possible, it is possible to make the effect on the attenuation, and consequently on the configuration for minimum attenuation, negligible.

However, if for any reason it is desired to support the conductors with respect to the shield in a manner which introduces a relatively large amount of leakage loss, it is still possible in many cases to determine the values of b/a and c/b for minimum attenuation. Thus it will be shown that the configuration which results in minimum attenuation is independent of the dielectric loss and the average dielectric constant, provided that the value of the average dielectric constant is independent of the configuration of the conductors and sheath.

At frequencies high enough so that ω^2L^2 is large compared with R^2 and ω^2C^2 is large compared with G^2 the attenuation of a transmission system in which the leakage loss is not negligible is

$$\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}} \qquad (51)$$

where G, the leakage loss, $=2\pi fpC$, p being the power factor and $C=k/L$.

Therefore, the high frequency attenuation is

$$\frac{R}{2}\sqrt{\frac{C}{L}}+\pi pf\sqrt{k} \qquad (52)$$

The last term in (52) is independent of the size and spacing of the conductors and sheath provided that the average dielectric constant k is independent of changes in configuration of the circuit. If this condition is fulfilled, the ratio of the inner diameter of the shield to the diameter of the conductors and the ratio of the interaxial separation of the conductors to the inner diameter of the shield for minimum attenuation for any given inner diameter of shield is independent of the dielectric loss: The high frequency characteristic impedance K is given closely by

$$\sqrt{L/C}$$

where C is the capacity per unit length of the system. This also is substantially independent of the dielectric loss.

These conditions would be met by completely filling the space between the conductors and shield with a material, such as oil or a solid rubber insulation or else by using insulators made of thin flat discs of an appropriate insulation, such as porcelain or glass. A suitable insulator might also consist of a solid dielectric, such as rubber in which air has been entrapped. If the air bubbles are small and evenly distributed through the non-gaseous portion of the dielectric, the average dielectric constant and the dielectric loss can be made nearly independent of the diameter and spacing ratios of the system.

If the value of the average dielectric constant varies with changes in the diameter and spacing ratios of the conductors and shield, it becomes extremely difficult to obtain a mathematical solution for the diameter and spacing ratios which give minimum attenuation for any fixed diameter of shield. However, when gaseous and non-gaseous dielectrics are arranged in any desirable manner, the ratio of the inner diameter of the shield to the diameter of the conductors and the ratio of the interaxial spacing of the conductors to the inner radius of the shield will not differ much from the values given by Fig. 9. If both non-gaseous and gaseous dielectrics are used, and are disposed in such a manner that the boundary surfaces between different dielectrics lie along followed by the flux in a homogeneous dielectric, the optimum configuration will apparently be the same as for a gaseous dielectric.

Such types of construction as those described above, involving the use of a partly or wholly non-gaseous dielectric may be desirable for either mechanical or electrical reasons.

The conditions that must be satisfied for maximum high frequency characteristic impedance for either pair of a shielded quad can also be determined. The high frequency characteristic impedance of a shielded solid pair when the conductors are small compared to the shield is:

$$K=\frac{4}{\sqrt{k}}\log\frac{1}{\lambda}\,\frac{1-\epsilon^2}{1+\epsilon^2} \qquad (53)$$

the same as when the second pair is absent. For any given ratio of inner diameter of shield to diameter of conductor the characteristic impedance becomes:

$$K=\frac{4}{\sqrt{k}}\log\text{ (constant)}+\frac{4}{\sqrt{k}}\log\left(\epsilon\frac{1-\epsilon^2}{1+\epsilon^2}\right) \qquad (54)$$

For maximum characteristic impedance the expression

$$\epsilon\frac{1-\epsilon^2}{1+\epsilon^2}$$

must be maximized. This can be accomplished by taking its derivative with respect to ϵ and putting it equal to zero. Thus we find that $\epsilon=.486$ for maximum characteristic impedance for any ratio of inner diameter of shield to diameter of conductor, providing the latter ratio is large.

If, however, the conductors are large with respect to the shield, Equation (53) no longer holds. However, the position of the conductors with respect to the shield must be such as to minimize the capacity, since the capacity and high frequency characteristic impedance are inversely proportional to each other. It can be seen that as the diameters of the conductors approach in size the inner radius of the shield, ϵ approaches .5 for minimum capacity and hence maximum high frequency characteristic impedance. Thus, for any given ratio of inner diameter of shield to diameter of conductor, the ratio of the interaxial separation of the conductors to the inner diameter of the shield should be between the limits .486 and .500. For practical purposes a value of about .49 may generally be used to secure maximum impedance.

It will be obvious that the general principles herein disclosed may be embodied in many other organizations widely different from those illustrated without departing from the spirit of the invention as defined in the following claims.

What is claimed is:

1. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, the relative dimen-

sions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

2. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another by a substantially gaseous dielectric, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

3. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, said conductors being helically twisted around the axis of said shield, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

4. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, said conductors being of such a type that conduction of currents whose frequencies are substantially above the audible range takes place substantially on the surface of said conductors, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

5. A transmission system comprising four solid cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

6. A transmission system comprising four hollow cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, said conductors having walls of substantial thickness as compared with their diameters, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

7. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another by a substantially gaseous dielectric, said conductors being of such a type that conduction of currents whose frequencies are substantially above the audible range takes place substantially on the surface of said conductors, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

8. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, said conductors being helically twisted around the axis of said shield, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, said conductors being of such a type that conduction of currents whose frequencies are substantially above the audible range takes place substantially on the surface of said conductors, the relative dimensions and spacings of said conductors and shield being such that the high frequency attenuation of said circuit will be a minimum.

9. A transmission system comprising four cylindrical conductors, a cylindrical conducting shield of predetermined diameter surrounding said conductors, one of said conductors being connected as a return for another to form a transmission circuit, said conductors and shield being insulated from one another, the ratio of the interaxial separation of said conductors to the inner diameter of said shield and the ratio of the inner diameter of said shield to the diameter of each of said conductors being such that the high frequency attenuation of the circuit will be a minimum.

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